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Applications of computer algebra to the study of recurrence in P-groups

Dikici, Ramazan

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APPLICATIONS OF COMPUTER ALGEBRA TO THE STUDY OF RECURRENCE IN P-GROUPS

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Ramazan Dikici

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Signature of Author ....R. Dikici..............

Ramazan Dikici

Summary

We have generalized the Wall-Vinson Theory of periodic sequences modulo a prime p to cover the 3–step Fibonacci case. In particular, we have proved that non-trivial short loops must be geometric. Indeed, this result is true in the additive group of the field $GF(p^n)$, where the relevant manipulations are better performed. We also demonstrated that the corresponding theorem fails to hold for the Fibonacci 4–step recurrence.

In another direction, we have generalized the Aydin-Smith theory of recurrences in finite p–groups. In particular, we have shown that, for the 3–step Fibonacci recurrence and any finite p–group of exponent p and nilpotency class 3, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3–step Fibonacci sequence in the field $GF(p)$. This involved doing the class 2 problem on the way. The corresponding theorem for class 4 is false.

We have addressed the question of recurrences with arbitrary coefficients. While we are not able to prove the corresponding theorem for either 2 or 3 step recurrences in general, we present a method for attacking individual questions. Given a specific recurrence, we believe that our method will prove a theorem of the form "For this particular recurrence, Wall's number behaves well in groups of exponent p having nilpotency class up to 3, except at finitely many bad primes". We have also presented experimental evidence that our method is likely to always produce a proof, though we can not guarantee this.

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Contents

Chapter 1

General Introduction

1.1 A Summary of This Thesis

This thesis contains material concerning the behaviour of recurrences in groups. It grows directly form Aydin's thesis [1], and considers two topics. The thesis is a compilation of three separate documents.

The first document is an extended version of a paper written by the author, jointly with H. Aydin and G. C. Smith, on 3-step recurrence relations in the integers modulo p where p is a rational prime [2]. The material on the three step Fibonacci sequence in this paper (chapter 2) was the work of R. Dikici, under the direction of his supervisor.

The second document (now chapter 3) is again an extended version of a Bath Technical Report [17] by Dikici and Smith. Here we investigate the finite $p-$quotients of groups like Fibonacci Groups [39]. These groups are manufactured from a recurrence relation in much the same way that the Fibonacci groups are defined. What we find is that the fundamental period of such a recurrence in a finite $p-$group of exponent p and nilpotency class 3 does not vary (usually) from the fundamental period of the analogous sequence in the integers modulo p. Such results can either be regarded as theorems about recurrences in finite $p-$groups, or as theorems about the $p-$quotients of these Fibonacci-like groups. The original report dealt only with the Fibonacci case, but we have extended it to cover general coefficients in the two and three step class 2 case.

The third document consists of the rest of the thesis, and is the work of R. Dikici under the direction of G. C. Smith. This is chapter 4. Here we extend the methods of chapter 3 to nilpotency class 3. We must make it clear that, in the case of general coefficients and nilpotency class 3, we are not claiming theorems. We claim to present a method for understanding the situation whenever we are challenged with a specific recurrence. The point is that the methods for the Fibonacci case have been generalized, but since there are infinitely many possible choices for the coefficients, and our analytical technique is not 'uniform', we do not have a general theorem. The real issue is that we do not know how to put into row echelon form a matrix of integers where the entries are variables.

The final section is an appendix containing computer programs which have been used to effect calculations. There is AXIOM [35] and CAYLEY [13] code which we hope is sufficiently self-explanatory to be readily understood by an interested reader. We refer in the thesis to the appendix when computational evidence is required.

1.2 Background Material

The purpose of this section is to give the definitions of some essential concepts and structures, and some important properties which will be used in the rest of this thesis. In addition to this a literature review will be given in the next section.

Definition 1.2.1 *A group G is a $p-$group if every element of G has order a power of a prime number p.*

The well-known result of $p-$groups theory is that the centre of a non-trivial finite $p-$group is non-trivial. All the $p-$groups have order p^n (for some n), and every finite $p-$group is nilpotent. Moreover, every non-trivial finite $p-$group has a central subgroup of order p. Groups of order p^n with $n > 2$ and nilpotency class exactly $n-1$ are known as $p-$groups of maximal class. Further information about $p-$groups can be found in [20] and [23] and a computer study of finite $p-$groups in [28].

Next we give the definition of a nilpotent group as they play an important rôle in this thesis.

Definition 1.2.2 *Let $H \triangleleft G$, $K \triangleleft G$, $K \leq H$. If H/K is contained in the centre of G/K then H/K is called a central factor of G. A group G is nilpotent if and only if it has a finite series of normal subgroups*

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_r = 1$$

such that G_{i-1}/G_i is a central factor of G for each i=1, 2, 3, $\cdots$, r.

The smallest value of r for any such central series of G is called the *class* of G. Thus abelian groups are the same as nilpotent groups of class 1, except that trivial groups are of class 0. If the nilpotency class of G is 2, then the group is metabelian.

Definition 1.2.3 *If x and y are elements of a group G, their commutator $x^{-1}y^{-1}xy$ is written (x, y). If X, Y are subgroups of G, then (X, Y) is the subgroup generated by all the commutators (x, y) with $x \in X$, $y \in Y$.*

By convention, for $n > 2$, $(x_1, x_2, \ldots, x_n) = ((x_1, x_2, \ldots, x_{n-1}), x_n)$ and similarly for subgroups. The concept of commutator arises from asking how near the elements x, y come to commuting. If any group G is nilpotent of class c, then every commutator $(x_1, \ldots, x_{c+1})$ is the identity, and, conversely, if every $(x_1, \ldots, x_{c+1}) = 1$, then G is nilpotent of class at most c. If G has nilpotency class c, then every subgroup and factor group of G has nilpotency class c. The readers are referred to [5], [19], [20] and [23] for more details concerning nilpotent groups.

We now discuss free groups. Roughly, a group F is said to be free if it has a subset X with the property that every element of F can be written uniquely is a product of elements of X and their inverses. This is made precise in the definition given below.

Definition 1.2.4 *A group F is said to be free on a subset $X \subseteq F$ if, given any group G and any map $\theta : X \longrightarrow G$, there is a unique homomorphism $\theta' : F \longrightarrow G$ extending θ, that is, having the property that $x\theta' = x\theta$ for all $x \in X$.*

Then X is called a *basis* of F and $|X|$ the *rank* of F. The well-known and most useful property of free group is that every group is isomorphic to a factor group of some free group.

Furthermore, F/R is called to be a *relatively free group* if F is a free group and R is a fully invariant subgroup of F.

Definition 1.2.5 *A group G is said to be cyclically presented if it has a presentation on n generators $a_1, a_2, \ldots, a_n$ with n relations obtained from a single word $w = w(a_1, a_2, \ldots, a_n)$ by permuting the subscripts modulo n according to the powers of the permutation $(12\ldots n)$.*

Finally, the *Fibonacci group* $F(r,n)$ is defined by the presentation

$$< x_1, x_2, \ldots, x_n : x_1x_2\ldots x_r = x_{r+1}, x_2x_3\ldots x_{r+1} = x_{r+2},$$

$$\ldots, x_{n-1}x_nx_1\ldots x_{r-2} = x_{r-1}, x_nx_1x_2\ldots x_{r-1} = x_r >,$$

where all subscripts are taken modulo n.

There have been various generalizations of the Fibonacci groups, for example the group $H(r,n,s)$ defined by the presentation

$$< x_1, x_2, \ldots, x_n : x_1x_2\ldots x_r = x_{r+1}x_{r+2}\ldots x_{r+s}, x_2x_3\ldots x_{r+1} = x_{r+2}x_{r+3}\ldots x_{r+s+1},$$

$$\ldots, x_{n-1}x_nx_1\ldots x_{r-2} = x_{r-1}x_r\ldots x_{r+s-2}, x_nx_1x_2\ldots x_{r-1} = x_rx_{r+1}\ldots x_{r+s-1} >,$$

where $r > s \geq 1$, and the groups $F(r,n,k)$ defined by the presentation

$$< x_1, x_2, \ldots, x_n : x_1x_2\ldots x_r = x_{r+k}, x_2x_3\ldots x_{r+1} = x_{r+k+1},$$

$$\ldots, x_{n-1}x_nx_1\ldots x_{r-2} = x_{r+k-2}, x_nx_1x_2\ldots x_{r-1} = x_{r+k-1} >,$$

where $r \geq 2$ and $k \geq 0$ and as usual all subscripts are taken to be reduced modulo n. Obviously, the groups $H(r,n,1)$ and $F(r,n,1)$ are each isomorphic to $F(r,n)$.

1.3 Literature Survey

The study of the Fibonacci groups $F(2,n)$ began with the question of Conway [15] as to whether or not $F(2,5)$ is cyclic of order 11. Soon after it was determined in [16] that this was indeed the case. Since then these groups have been a subject of interest for a large number of mathematicians.

Wall was concerned with determining the length of the period of the recurring series obtained by reducing a Fibonacci series by a modulus m and conjectured that $k(p^2) = k(p)$, where k denotes the length of the shortest period and p is a prime number. He verified on the base of computer search that his conjecture was true for primes less than 10^4 [41].

Vinson as another early contributor interested in the problem in a slightly different perspective by working on the rank of apparition [40].

The main problem is to decide when $F(r,n)$ is finite and, if it is, to determine its structure. In the mid-seventies it was shown that derived factor groups $F(r,n)/F'(r,n)$ are finite and a formulae for their orders was given in [22].

A criterion was given for $F(r,n)$ to be infinite which depends on the fact that any finite $F(r,n)$ has trivial multiplicator and invokes Golod-Šafarevič theorem and also it was shown that, if n is a divisor of r, then $F(r,n)$ is cyclic of order $r-1$ [25]. This has been one of the substantial papers in this area.

The Fibonacci group $F(2,7)$ was known to be cyclic of order 29 by computer coset enumeration, which exhibits only the result, though Havas gives an algebraic proof [21]. $F(2,8)$ and $F(2,10)$ were shown to be infinite in [6] and $F(2,9)$ was recently shown to be infinite by Newman in [29]. An unpublished result is that of Lyndon [27], who used small cancellation theory to show that $F(2,n)$ is infinite for $n \geq 11$, a major contribution which almost finished the problem of the $F(2,n)$. A generalization of Lyndon's proof to arbitrary r has been given in [14] by applying the elegant and powerful methods of small cancellation theory to these groups.

Though we know exactly when $F(2,n)$ is infinite, we do not have a complete theory for $F(r,n)$. Thanks to the work of Thomas [37] in the form of a technical report (which appears more widely in [39]) what is known about $F(r,n)$ up to date is readily accessible.

Most of the groups of Fibonacci type which are known to be finite are metacyclic. The Todd-Coxeter coset enumeration algorithm has been used to discover a finite non-metacyclic Fibonacci group $F(3,6)$ and to determine its structure [8]. Another non-metacyclic group of order 1512 not isomorphic to $F(3,6)$, in spite of them having the same order, has been given in [10]. Furthermore it was shown that, if $r \equiv 1 \pmod n$ and $r > 1$, then $F(r,n)$ is metacyclic of order $r^n - 1$ [9] [11].

The Fibonacci groups have continuously attracted the attention of mathematicians in the eighties. Seal [33] determined the orders of the groups $F(r,3)$ for $r \equiv 2 \pmod 3$ and $F(r,4)$ for $r \equiv 2 \pmod 4$ and proved that various other Fibonacci groups are infinite by using the method similar to those in [14].

Some results of [25], generalized in [12] and [36], give for example that $F(r,n)$ is infinite if $(r+1,n) > 3$ or if $(r+1,n) = 3$ with n even or $r > 2$.

The most prolific co-operation between Campbell, Robertson and Thomas give us some insight into groups which have presentations closely related to those of Fibonacci groups. Thomas has proved that the Fibonacci groups $F(4k+2,4)$ are metacyclic for all k, giving an affirmative answer to a conjecture of Seal, and also he shows that $F(4k+2,4)$ is isomorphic to the generalized Fibonacci groups $H(4k+3,4,2)$ [38].

The recent work in this area has been to investigate which groups $F(2,n)$ have a given finite quotient G of a particular form. The case when G is cyclic of prime order has been studied by Wall [41], Vinson [40] and Wilcox [42]. The results of [41] and [40] concerning the divisibility properties of the Fibonacci sequences has been improved to the general Lucas sequence [34]. Campbell, Doostie and Robertson have addressed the similar questions where the image group is simple [7]. Doostie somewhat overlaps this work in his Ph.D thesis [18].

Aydin has generalized the Wall-Vinson theory to address the case where the image group is an arbitrary finite p−group and proved that, if the Fibonacci group $F(2,n)$ has the two generator relatively free group in the variety of exponent p groups of class 1 as a homomorphic image, then $F(2,n)$ has the two generator relatively free group G in the variety of exponent p groups of class 4 as a homomorphic image [1].

Chapter 2

Wall and Vinson Revisited

2.1 Introduction

Let G be a finite group. We define a family of recurrences, parameterized by natural numbers n, via equations

$$(R_n) \qquad x_i = x_{i-1}x_{i-2}\cdots x_{i-n}.$$

We fix $n > 1$. We may 'start' the recurrence with any initial data $g_0, g_1, \ldots, g_{n-1}$ and then the recurrence will define a periodic bi-infinite sequence (g_i) indexed by the integers. We use the term *loop* to describe a bi-infinite sequence satisfying the recurrence. Notice that, if $\alpha \in \mathbf{Z}$, then putting $h_i = g_{i+\alpha}$ for $0 \leq i < n$ gives another set of initial data. The recurrence R_n then gives us another loop (h_i) of G. Notice that (h_i) is just (g_i) shifted through α steps. We say that (h_i) is a *rotation* of (g_i).

The fundamental period of a loop is clearly a significant number, and we call it *Wall's number* after a pioneer in this area. Following Wall [41], we shall use the letter k (suitably adorned) to describe this period. We shall clarify this point later. In all fairness, we should really cite Lucas rather than Wall, but the term *Lucas Number* is already in use. The reader's attention is drawn to [26], [31] and [32].

Wall investigated the recurrence R_2 in finite cyclic groups. This subject immediately reduces to the study of cyclic groups of prime power order. This paradigm is exploited elsewhere [1, 3, 4] to study recurrences in finite nilpotent groups, since they are the cartesian product of their Sylow p-subgroups. Another early contributor to this field was Vinson [40], who was particularly interested in ranks of apparition, a mistranslation from the French. More interest has come in the form of [7, 18, 34, 39] and [42].

There have been exciting recent developments in this area. Pinch [30] has studied the relationship between the period of a general linear recurrence modulo a rational prime p, and the period modulo a power of that prime. He does this via examining the algebraic number theory of certain finite extensions of the p-adic numbers.

Wall distinguishes the special loop $s = (s_i)$ with initial data $s_0 = 0$ and $s_1 = 1$ in $\mathbf{Z}/p^n\mathbf{Z}$. Let $k(s, p^n)$ denote the fundamental period of s.

Theorem 2.1.1 *(D.D. Wall [41]) The number $k(s,p^n)$ divides $k(s,p)p^{n-1}$, and the two quantities are equal provided $k(s,p) \neq k(s,p^2)$.*

Wall goes on to conjecture that, for all primes p, we always have $k(s,p) \neq k(s,p^2)$. He announced that he had verified this result for all primes $p < 10^4$. We have tested the conjecture on more modern computers, and can confirm that the result holds for all primes less than 10^8. We include the code in Appendix A, in case the reader is interested in chasing this still further.

The theorem 2.1.1 has recently been generalized. A slightly weaker version of it is now known to hold in any finite p-group P for arbitrary t-step linear recurrences [4]. The rôle of n in this context is the minimum length of an exponent p central series of P.

In Appendix B we include some mysticism; an unjustifiable probabilistic argument to persuade the gullible that Wall's conjecture is true.

Let us consider the more general recurrences written multiplicatively as

$$x_i = x_{i-1}^{a_1} x_{i-2}^{a_2} \cdots x_{i-n}^{a_n},$$

or additively as

$$(A_n) \qquad x_i = \sum_{r=1}^{n} a_r x_{i-r}.$$

The exponents/coefficients are deemed to be integers. Such a recurrence may not be so well behaved as R_n, for it may be singular in a particular group. This means that initial data may not define a unique bi-infinite periodic sequence; the problem arises when attempting to extend the sequence through negative integer subscripts. Certainly all is well if $a_n \in \{1, -1\}$. Equally well, if we are working in a p−group all is well provided $gcd(p, a_n) = 1$. We shall assume, from now on, that all our sequences are non-singular in the groups to which we apply them. This important *caveat* should be noted, since we do not intend to adjoin non-singularity conditions repeatedly to our results.

2.2 Two Step Recurrences

The theory of general two-step recurrences is very well understood; see [26], and chapter 2 of [32]. The following results of this section are well-known [40, 41], but we prove them from our perspective as a prelude to section 3.

Theorem 2.2.1 *Let H be the additive group of a finite field $K = GF(p^t)$. Consider the standard sequence $s = (s_i)$ for the recurrence A_2. Let $k(s)$ denote the Wall number of s. Suppose $b = (b_i)$ is any loop satisfying A_2 in H; then $k(b)$ divides $k(s)$. If $k(b) < k(s)$, then b must either be the trivial loop (all entries 0) or must be a geometric sequence.*

Proof Let s^+ denote the rotation of s through one step. Thus $s_0^+ = 1$ and $s_1^+ = a_1$. Now $b = b_0 \cdot s^+ + (b_1 - a_1 b_0) \cdot s$. Thus $k(b)$ divides $k(s)$. Now suppose b is not the trivial loop (so that at least one entry of b is nonzero), but that at least one entry if b does vanish. Multiplying through by a scalar and rotating if necessary we find that $k(b) = k(s)$.

Thus any short loop must be trivial, or contain no entry which is 0. Suppose that b is of the latter form. Choose $i \in \mathbf{Z}$ and let b^{+i} be the rotation of b through i steps, so $b_0^{+i} = b_i$ and $b_1^{+i} = b_{i+1}$. Put $c = b_0 b_i^{-1} \cdot b$. Thus $k(c) = k(b)$. Now $k(c - b)$ must divide $k(b)$ and so $d = c - b$ is a short loop. However, $d_0 = 0$ so d is the trivial loop. Thus $b_1 = b_0 b_i^{-1} b_{i+1}$. We conclude that $b_i^{-1} b_{i+1} = \gamma$ is independent of i so b is a geometric progression with common ratio γ. The ratio γ must be a root of $X^2 - a_1 X - a_2$.

Corollary 2.2.2 *Let $\Delta = a_1^2 + 4a_2$. If Δ is not a square in K then there can be no non-trivial short loops.*

Theorem 2.2.3 *Once again we work in the additive group H of a finite field $GF(p^t)$. At most one of the roots of $f = X^2 - a_1X - 1$ can give rise to a short geometric loop (assuming $p \not| \Delta$).*

Proof Let β_1, β_2 be roots of f, possibly in a splitting field. These roots give rise to loops $(1, \beta_i, \beta_i^2 \ldots)$. The difference of these loops contains a 0 and is non-trivial, so $lcm(o(\beta_1), o(\beta_2)) = k$. If f fails to split in $GF(p)$ then β_1, β_2 are conjugate under the action of $Gal_{GF(p)}GF(p^2)$ and so must have the same order, which must therefore be k. Conversely, if f splits over $GF(p)$, then we assume that $o(\beta_1) \leq o(\beta_2)$. Suppose that $o(\beta_1) = n$. Thus $\beta_2^n = (-1)^n$. If n is even then both geometric loops must have length k. We may therefore assume that $\beta_2^n = -1$ so $o(\beta_2)$ divides $2n$ and is greater than n. Thus $o(\beta_2) = 2n$. Now $k = lcm(o(\beta_1), (\beta_2)) = 2n$ so β_2 does not give rise to a short loop.

Observation If we replace -1 by $+1$ in the theorem above then any geometric loop must have length k.

Application We examine the two step Fibonacci recurrence studied by Wall. In this case the field K is $GF(p)$ and $\Delta = 5$. Thus there are no short loops if $(5/p) = -1$. We are using the Legendre symbol defined as

$$(a/p) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue (mod } p) \\ -1 & \text{if } a \text{ is a quadratic non-residue (mod } p). \end{cases}$$

By quadratic reciprocity, if $p \neq 2, 5$ then $(5/p) = (p/5)$. Thus, save for the excluded primes, there are no short loops when $p \equiv 2$ or 3 mod 5.

Theorem 2.2.4 *In $GF(p), k \mid p^2 - 1$ providing $p \not| \Delta$.*

Proof Working in $GF(p^2)$ if necessary (to split $X^2 - a_1X - a_2$) we obtain two distinct roots of this polynomial γ_1, γ_2, yielding two geometric loops $r(1), r(2)$ respectively. Here $r(j)_0 = 1$ and $r(j)_1 = \gamma_j$ for $j \in \{1, 2\}$. Now put $\lambda = (\gamma_1 - \gamma_2)^{-1}$, then $s = \lambda \cdot (r(1) - r(2))$. Thus k divides $lcm(o(\gamma_1), o(\gamma_2))$. The multiplicative order of any non-zero element of $GF(p^2)$ divides $p^2 - 1$, and so we are done.

Returning to our application, we have the special case that $\gamma_1\gamma_2 = -1$, so their multiplicative orders differ by a factor of 2. A little analysis enables us to recover Wall's result that $k \mid p-1$ if $p \equiv 1,4 \bmod 5$ and $k \mid 2p+2$ otherwise (save when $p = 2$ or 5.) In the former case, $X^2 - X - 1$ splits in $GF(p)$, in the latter case it does not, but splits in $GF(p^2)$.

We point out some connections with *Fermat's Last Theorem.* There is Weiferich's result that, if the first case of Fermat's Last Theorem is false for the prime exponent p, then

$$2^{p-1} \equiv 1 \bmod p^2.$$

There is also Mirimanoff's extension of this result that it would also follow that

$$3^{p-1} \equiv 1 \bmod p^2.$$

If Wall's conjecture were false for a prime $p \equiv 1$ or 4 mod 5 then a golden ratio γ would have to satisfy a congruence of Weiferich-Mirimanoff type. For a discussion of these congruences, see [32] 166-169.

2.3 Higher Step Recurrences

Here we believe that our results may be new. We now press on to discuss the recurrences R_3 and R_4. We work in the additive group H of a finite field as before. We will find that the analysis of this case is remarkably similar to the case R_2, but that, in moving up to R_4, the situation becomes more complex.

Let the standard loop s begin (0,0,1), and let its period be $k_3(p)$, written as k in this section. As before, the period of any loop must divide k for the usual reasons.

Theorem 2.3.1 *Short loops must be geometric for the 3-step Fibonacci recurrence in H, the additive group of the finite field $GF(p^n)$.*

Proof We tackle this proof via a sequence of lemmas. Let f be any non-trivial loop containing two consecutive zeros. It must be a non-zero scalar multiple of a rotation of the standard loop and so has period k. Now suppose that g is a non-trivial loop containing at most one consecutive zero. By rotation and scaling we may assume that g begins $0, 1, \alpha$ for some $\alpha \in H$.

Let the roots of $X^3 - X^2 - X - 1$ be τ_1, τ_2 and τ_3. The discriminant of $X^3 - X^2 - X - 1$ is 44, so, in what follows, we ban the primes 2 and 11 as being "bad", and analyze them separately. The polynomial has repeated roots for those primes and there are short loops with 0, so they must be exceptional primes. Thus τ_1, τ_2 and τ_3 are distinct and $\tau_1\tau_2\tau_3 = 1$. We distinguish six special loops ($j \in \{1, 2, 3\}$),

$$r(j) = (1, \tau_j, \tau_j^2, \ldots)$$

and

$$h(j) = (0, 1, 1 - \tau_j, \ldots).$$

Lemma 2.3.2 *If $i \neq j$ then $lcm(k(h_i), k(h_j)) = k$.*

Proof $h_i - h_j = (\tau_j - \tau_i) \cdot s$ where s is the standard loop. Thus $k(h_i - h_j) = k$, so that $lcm(k(h_i), k(h_j)) = k$.

Lemma 2.3.3 *If i, j and m are distinct then $k(h_m) \mid lcm(k(r(i)), k(r(j)))$.*

Proof To see this, notice that $r(i) - r(j) = (\tau_i - \tau_j)h_m$ – here we are using the fact that $\tau_1 + \tau_2 + \tau_3 = 1$. Thus $k(h_m) \mid lcm(k(r(i), k(r(j))$.

Lemma 2.3.4 *If $i \neq j$ then $lcm(k(r_1), k(r_2), k(r_3)) = lcm(k(r(i), k(r(j))))$.*

Proof Let

$$z = lcm(o(\tau_1), o(\tau_2)) = lcm(k(r(1)), k(r(2))).$$

Now raising the equation $\tau_1\tau_2\tau_3 = 1$ to a power we obtain $\tau_1^z\tau_2^z\tau_3^z = 1$. Thus $o(\tau_3) \mid z$, so that

$$lcm(o(\tau_1), o(\tau_2), o(\tau_3)) = z.$$

The choice of τ_1 and τ_2 was arbitrary and so our claim is justified.

Lemma 2.3.5 *If $i \neq j$ then $lcm(k(r(i)), k(r(j))) = k$.*

Proof By 2.3.2 we have

$$k = lcm(lcm(k(h_2), k(h_3)), lcm(k(h_1), k(h_3))) = lcm(k(h_1), k(h_2), k(h_3)).$$

Now we deploy 2.3.3 to deduce that

$$lcm(k(h_1), k(h_2), k(h_3)) \mid lcm(k(r(1)), k(r(2)), k(r(3))).$$

Finally we deploy 2.3.4 to obtain that k divides $lcm(k(r(i)), k(r(j)))$ whenever i and j are distinct. However $k(r(i))$ and $k(r(j))$ must be divisors of k and so we are done.

Lemma 2.3.6 *If i, j, m are distinct then $lcm(k(r(i)), k(r(j))) \mid k(h_m)$.*

Proof To see this, notice that $r(i) - r(j) = (\tau_i - \tau_j)(0, 1, 1 - \tau_m \ldots)$ and so has length $k(h_m)$. Now $\tau_i^{k(h_m)} - \tau_j^{k(h_m)} = 0$ and $\tau_i^{k(h_m)+1} - \tau_j^{k(h_m)+1} = \tau_i - \tau_j$. Subtracting we obtain $\tau_i^{k(h_m)} = 1$. Thus $k(r(i)) \mid k(h_m)$ whenever i and m are distinct as required.

Piecing together 2.3.3, 2.3.5 and 2.3.6, we have established that $k(h_i) = k$ for all $i \in \{1, 2, 3\}$. It remains to address loops of the form $(0, 1, 1 - \alpha, \ldots)$ where α is *not* a root of $X^3 - X^2 - X - 1$. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1-\alpha \\ 1 & 1-\alpha & 2-\alpha \\ 1-\alpha & 2-\alpha & 4-2\alpha \end{pmatrix}.$$

More matrices are useful. We define them as

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

and finally

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Notice that $A = B - \alpha C = (F - \alpha I)C$, so that $det(A) = det(F - \alpha I)(-1)^3 = det(\alpha I - F) = \alpha^3 - \alpha^2 - \alpha - 1$. However, the choice of α forces $det(A) \neq 0$. The fact that A is non-singular enables us to write the standard loop s as a linear combination of rotations of $(0, 1, 1 - \alpha, \ldots)$. Thus such a loop $(0, 1, 1 - \alpha, \ldots)$ must have period k.

We conclude that any non-trivial short loop must contain no entry which is 0. By the same argument used in the two step case it follows that any candidate short loops must be geometric progressions with common ratio a root of $X^3 - X^2 - X - 1$ and we have proved 2.3.1.

The situation here is more complex than in the two step case; it is possible for all three roots to give rise to genuine short loops. We shall call these loops *golden loops.* The first few instances of this phenomenon we outline here.

Prime	Golden Loop Length	Wall Number
4481	2240	4480
	640	
	896	
4621	84	4620
	660	
	2310	
6007	182	2002
	154	
	1001	
6917	364	6916

	532	
	3458	
14281	595	4760
	680	
	952	

Passing to the Fibonacci 4-step recurrence the situation gets less pleasant. It is possible for non-trivial short loops to have entries which vanish. The prime 7 is a case in point. Wall number is 342 in $GF(7)$ but there is a short loop containing a 0 of length 171.

2.4 Appendix A: Computer Search

We sought a counterexample to Wall's conjecture, and failed to find one less than 10^8. We ran programs for several weeks in the background on a variety of SUN 3, SUN 4 and Orion machines. Each machine examined a congruence class or congruence classes of primes modulo 30, depending on the machine's relative speed. For almost all of the calculation, the arbitrary precision arithmetic feature of CAYLEY [13] was used to perform the computation, though as an experiment J P Fitch coded the algorithm in LISP and eliminated a small part of the range (circa 5 million).

Note that the repeated squaring of the Fibonacci matrix can be performed more easily than squaring an arbitrary matrix, since all powers of the matrix are symmetric.

The code follows:

```
x = 1;
''Initialize the Fibonacci Matrix''
a11 = 1;
a12 = 1;
a22 = 0;

a = seq(a11,a12,a22);
'' Procedure for multiplying matrices of our
   special form mod q''
procedure mult(a,b,q;c);
        c11 = (a[1]*b[1] + a[2]*b[2]) mod q;
        c12 = (a[1]*b[2] + a[2]*b[3]) mod q;
        c22 = (a[2]*b[2] + a[3]*b[3]) mod q;
        c = seq(c11,c12,c22);
end;

''Procedure writes the integer q into a reverse
  binary sequence''
procedure decomp(q;bin);
        bin = empty;
        while q ne 0 do
                r = q mod 2;
                q = (q-r)/2;
                bin = append(bin,r);
        end;
end;

''Main Program''
zz = 50000000 mod 30;
for i = (50000000 - zz + x) to 100000000 by 30 do
```

```
        if i mod 1000000 lt 30 then
                print ' done up to ',i;
        end;
        if not prime(i) then
                loop;
        end;
        y = i^2;
        ''Split into two cases; determine the
appropriate multiple of Walls Number.''
        if ((i mod 5) eq 1) or ((i mod 5) eq 4) then
                k = i-1;
        else k = 2*i+2;
        end;
        decomp(k;bin);
        u= seq(1,0,1);
        ustore = u;
        m = conseq(u,length(bin));
        m[1] = a;
        b = a;
        ''Compute the appropriate power of the
       Fibonacci matrix mod y
          using the method of repeated squaring.'';
        for j = 2 to length(m) do
                mult(b,b,y;b);
                m[j] = b;
        end;
        for j = 1 to length(m) do
                if bin[j] eq 1 then
                        mult(u,m[j],y;u);
                end;
```

```
        end;
        if u eq ustore then
                print '*violator*',i;
        end;
end;
print ' done ';
show time;
```

2.5 Appendix B: Mysticism

The Euler product form of the Riemann function is

$$\zeta(s) = \Pi_p(1 - p^{-s})^{-1}$$

where the product is taken over all primes p. The more conventional expression is $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Recall Euler's result that $\zeta(2) = \pi^2/6$.

Now for the nonsense. When calculating k mod p, we seek the first repetition of the adjacent pair $(0,1)$ as (s_k, s_{k+1}) in the standard sequence. Modulo p^2 there are p choices the value of s_k and p choices for the value of s_{k+1}. Now pretend that these values occur at random, independently, and with a uniform probability distribution. (Do not tell the children!) Now the probability that, for no prime p, $(s_k, s_{k+1}) = (0,1)$ mod p^2 is

$$\Pi_p(p^2 - 1)/p^2 = \Pi_p(1 - p^{-2}) = \zeta(2)^{-1} = 6/\pi^2.$$

Thus the probability that Wall's conjecture is true is a little less than 2/3. However, let p_x denote the probability that Wall's conjecture holds for all primes $p > x \in \mathbf{R}$. On the basis of our (preposterous) assumptions, we obtain

$$p_x = \Pi_{p>x}(p^2 - 1)/p^2 = \Pi_{p>x}(1 - p^{-2}).$$

Let us compute some values of p_x. We obtain

$$p_3 = (\zeta(2)(1-2^{-2})(1-3^{-2}))^{-1} = 0.91189,$$

and by similar arguments, $p_{100} = 0.99818$, $p_{1000} = 0.99987$ and $p_{10000} > 0.99999$.

Since Wall himself knew that his conjecture was true for primes less than 10^4, and the probability that it fails for any larger prime is a little less than 0.00001, one might be persuaded that Wall's conjecture is likely to be true. We acknowledge the Computer Algebra system AXIOM (ex-SCRATCHPAD) [35] which was used to perform this calculation.

Chapter 3

Fibonacci Sequences in Groups

3.1 Introduction

The problem of understanding linear recurrences in groups has recently attracted some attention. Thomas [39] gives an excellent survey, and Campbell, Doostie and Robertson [7] have attacked the problem of recurrences in the case of non-abelian finite simple groups. There is also a forthcoming paper [24] of Johnson and Odoni which characterizes when a group of Fibonacci type is cyclic when abelianized.

For the integers modulo n, the classical paper is [26], though some interesting points were made in [41]. The text [32] gives easy access to Lucas' ideas. Further impetus to this subject is given by Pinch [30] who has used the methods of algebraic number theory to clarify the position. Entirely different methods are used in [2], which specifically concentrates on 3−step recurrences. This work is an updated version of [17] and is a sequel to [3].

Let (s_i) denote the standard Fibonacci (sometimes *Tribonacci*) sequence in $GF(p)$ defined by the recurrence $s_{i+3} = s_i + s_{i+1} + s_{i+2}$ and the initial data $s_0 = 0,\ s_1 = 0$ and $s_2 = 1$. This sequence (or loop) must be periodic, and we use the letter k to denote the fundamental period of (s_i). Thus k depends on p.

One can define sequences by a linear recurrence in any group. As has been pointed out elsewhere [1, 3, 4], the fundamental period of such a sequence in a finite p−group P is closely related to the fundamental period of the standard sequence (corresponding to that recurrence) in $GF(p)$. In this section, we show that, for a particular class of $p-$groups, and a particular recurrence, the connection with k is very direct. The fundamental period of a sequence (satisfying a linear recurrence) is sometimes called the *Wall Number* of that sequence. The Wall Number (with respect to a particular linear recurrence) $k(G)$ of a group G the lowest common multiple of the fundamental period of all sequences in the group G which satisfy the recurrence.

We consider a 3−step Fibonacci sequence $r = (r_i)$ in a finite $p-$group G, given some initial data r_0, r_1 and r_2. Such a sequence (or loop) must be periodic and we denote the fundamental period by k_r. By [4] we know that k_r must divide kp^n for some $n \in \mathbf{N}$.

In general we must deploy a Computer Algebra system to complete certain calculations in the course of our analysis of the recurrence though, for the purposes of this exposition, we have included many 'hand calculations' which could have been performed in a trice using a Computer Algebra system. At one point this is impractical, and we use AXIOM [35] to compute a 12 by 12 determinant. We hope that the reader will regard this slightly long-winded approach as illuminating. A compact version of the theory will appear in a future paper of Dikici and Smith.

When the recurrence is more than 3−step, we shall definitely have recourse to computational assistance since the calculations can easily exceed the attention span of the reader – who might well not be overly interested in the details of putting a 500 by 800 matrix in row echelon form.

We must acknowledge CAYLEY [13], using which many of the results in this area were originally discovered as 'experimental truths'.

3.2 Result

Recall that for the purposes of this section, k denotes the fundamental period of the standard 3−step Fibonacci sequence $0, 0, 1, 1, 2, \ldots$ taken modulo a distinguished prime p.

Theorem 3.2.1 *Let $p > 3$ be a prime number; if G is a non-trivial finite $p-$group of exponent p and nilpotency class at most 3 then $k(G) = k$.*

Of course, if G is the trivial group, then $k(G) = 1$.

3.3 Proof: Nilpotency Class 2

Let $p > 3$ be a prime number. Consider the 3-generator relatively free group G in the variety of nilpotent groups of class 2 and exponent p. We suppose G is free on a set of generators $\{g_1, g_2, g_3\}$. This group G has order p^6 and we put $(g_2, g_1) = g_4$, $(g_3, g_1) = g_5$ and $(g_3, g_2) = g_6$. The subgroup $< g_4, g_5, g_6 >$ has order p^3, and is both the centre and derived group of G. Every element of G has a unique representation as

$$g_1^a g_2^b g_3^c g_4^d g_5^e g_6^f,$$

where the exponents are elements of $GF(p)$. Having distinguished this way of writing elements we can even think of group elements as vectors of dimension 6 over GF(p), i.e. as (a, b, c, d, e, f).

We need a formula for the product of three elements of the group. Suppose

$$(a_0, b_0, c_0, d_0, e_0, f_0) \cdot (a_1, b_1, c_1, d_1, e_1, f_1) \cdot (a_2, b_2, c_2, d_2, e_2, f_2) = (a_3, b_3, c_3, d_3, e_3, f_3)$$

then there are polynomial formulas for $(a_3, b_3, c_3, d_3, e_3, f_3)$. Explicitly they are

$$a_3 = a_0 + a_1 + a_2,$$

$$b_3 = b_0 + b_1 + b_2,$$

$$c_3 = c_0 + c_1 + c_2,$$

$$d_3 = d_0 + d_1 + d_2 + a_1 b_0 + a_2(b_0 + b_1),$$

$$e_3 = e_0 + e_1 + e_2 + c_0 a_1 + (c_0 + c_1) a_2$$

and

$$f_3 = f_0 + f_1 + f_2 + c_0 b_1 + (c_0 + c_1) b_2.$$

Define a bi-infinite sequence $(r_i) = ((a_i, b_i, c_i, d_i, e_i, f_i))$ using the 3−step Fibonacci recurrence and initial data

$$r_0 = (1,0,0,0,0,0),$$

$$r_1 = (0,1,0,0,0,0)$$

and

$$r_2 = (0,0,1,0,0,0).$$

We define some auxiliary three step Fibonacci sequences $(s_i), (u_i)$ and (t_i) by using the following initial data:

$$(s_0, s_1, s_2) = (0,0,1),$$

$$(t_0, t_1, t_2) = (0,1,0)$$

and

$$(u_0, u_1, u_2) = (1,0,0),$$

so $(a_i) = (u_i)$, $(b_i) = (t_i)$ and $(c_i) = (s_i)$. Notice that both (t_i) and (u_i) can be expressed in terms of (s_i). Specifically we have, for each i, the pair of equations

$$t_i = s_{i+1} - s_i \tag{3.1}$$

and

$$u_i = s_{i+2} - s_{i+1} - s_i. \tag{3.2}$$

Using the triple product multiplication rule, we may also write down formulas for d_α, e_α and f_α in terms of elements of the sequences (s_i), (t_i) and (u_i). A straightforward induction yields that for $\alpha \geq 0$ we have

$$d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(u_{i+1}t_i + u_{i+2}(t_i + t_{i+1})),$$

$$e_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i u_{i+1} + u_{i+2}(s_i + s_{i+1})),$$

and

$$f_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1})).$$

We seek to show that the Wall number for the sequence $(a_i, b_i, c_i, d_i, e_i, f_i)$ is actually k (the Wall number for (s_i) in $GF(p)$). In order to do this we must show that

$$d_k = d_{k+1} = d_{k+2} = e_k = e_{k+1} = e_{k+2} = f_k = f_{k+1} = f_{k+2} = 0. \tag{3.3}$$

Everything may be expressed in terms of the sequence (s_i), so we first study that sequence. From now on we will use the notation $\sum$ to denote a sum taken over one fundamental period where the variable of summation is clear – it will usually be i or j.

Lemma 3.3.1 *The following equations associated with the standard* $3-$ *step Fibonacci sequence, all hold.*

$$\sum s_i = 0 \tag{3.4}$$

$$\sum s_i s_{i-1} = 0 \tag{3.5}$$

$$\sum s_i^2 = 0 \tag{3.6}$$

Proof This starts off very simply; we have

$$\sum s_i = \sum s_{i-1} + \sum s_{i-2} + \sum s_{i-3} = 3 \sum s_i,$$

and since 2 is a prohibited prime, (3.4) is established.

Now observe that

$$\sum s_i^2 = \sum (s_{i+2} - s_{i+1} - s_{i-1})^2$$

so

$$\sum s_i^2 = 3\sum s_i^2 - 2\sum s_{i+2}s_{i+1} - 2\sum s_{i+2}s_{i-1} + 2\sum s_{i+1}s_{i-1}.$$

Use the relation $s_{i+2} = s_{i+1} + s_i + s_{i-1}$ in the third sum to obtain

$$\sum s_i^2 = 3\sum s_i^2 - 2\sum s_{i+2}s_{i+1} - 2\sum s_{i+1}s_{i-1} - 2\sum s_i s_{i-1} - 2\sum s_{i-1}^2 + 2\sum s_{i+1}s_{i-1},$$

so that

$$\sum s_i^2 = \sum s_i^2 - 4\sum s_i s_{i-1},$$

and thus $\sum s_i s_{i-1} = 0$. We have established (3.5) (again noting that 2 is a prohibited prime).

Finally we have

$$\sum s_i^2 = \sum (s_{i-1} + s_{i-2} + s_{i-3})^2,$$

so that

$$\sum s_i^2 = 3\sum s_i^2 + 2\sum s_{i-1}s_{i-2} + 2\sum s_{i-1}s_{i-3} + 2\sum s_{i-2}s_{i-3},$$

and therefore

$$\sum s_i^2 = 3\sum s_i^2 + 4\sum s_{i-1}s_{i-2} + 2\sum s_{i-1}s_{i-3}.$$

Thus

$$\sum s_i^2 = 3\sum s_i^2 + 4\sum s_{i-1}s_{i-2} + 2\sum s_{i-2}s_{i-3} + 2\sum s_{i-3}^2 + 2\sum s_{i-3}s_{i-4},$$

so that

$$\sum s_i^2 = 5\sum s_i^2 + 8\sum s_{i-1}s_{i-2} = 5\sum s_i^2.$$

We deduce that $\sum s_i^2$ vanishes and (3.6) is justified. Once again we have used the fact that $p \neq 2$.

From now on we will apply the hypothesis that $p > 3$ without comment. The next corollary is obtained from (3.5) and (3.6) using the Fibonacci recurrence and finite induction.

Corollary 3.3.2 *For all integers α and β we have*

$$\sum s_{j+\alpha}s_{j+\beta} = 0. \tag{3.7}$$

Now back to task at hand. We define

$$c_{a,b} = \sum s_i s_{i+a} s_{-i+b}$$

where $a, b \in \mathbf{Z}$, and we shall show that, for all a and b, we have $c_{a,b} = 0$. We initially observe that the quantities $c_{a,b}$ satisfy the recurrences

$$c_{a,b} = c_{a,b-1} + c_{a,b-2} + c_{a,b-3}, \tag{3.8}$$

$$c_{a,b} = c_{a-1,b} + c_{a-2,b} + c_{a-3,b} \tag{3.9}$$

$$and c_{a,b} = c_{a+1,b-1} + c_{a+2,b-2} + c_{a+3,b-3}. \tag{3.10}$$

These recurrences are not quite enough to force each $c_{a,b}$ to vanish. We need another, less obvious, system of equations.

Lemma 3.3.3 *For all integers α and β we have*

$$s_{\alpha+\beta} = s_\alpha s_{\beta+2} + (s_{\alpha+1} - s_\alpha)s_{\beta+1} + (s_{\alpha+2} - s_{\alpha+1} - s_\alpha)s_\beta. \tag{3.11}$$

Proof Consider the rotation of the standard 3-step Fibonacci sequence

$$s^{+\alpha} = (s_\alpha, s_{\alpha+1}, s_{\alpha+2}, \ldots).$$

We can express this sequence as a linear combination of s, s^+ $(= s^{+1})$ and s^{++} $(= s^{+2})$ by

$$s^{+\alpha} = s_\alpha s^{++} + (s_{\alpha+1} - s_\alpha)s^+ + (s_{\alpha+2} - s_{\alpha+1} - s_\alpha)s. \tag{3.12}$$

To see this, just observe that the sequences on each side of this equation satisfy the 3-step Fibonacci recurrence, and agree in positions $0, 1$ and 2. Now take the entry in position β on each side to obtain the desired formula (3.11), and we are done.

Lemma 3.3.4 *The quantities $c_{a,b}$ satisfy the equation*

$$c_{a,b+2} + c_{a+1,b+1} - c_{a,b+1} + c_{a+2,b} - c_{a+1,b} - c_{a,b} = 0 \tag{3.13}$$

for all integers a and b.

Proof Put $\alpha = i + a$ and $\beta = -i + b$ in (3.11), multiply through by s_i and sum over the range $0 \leq i < k$ to obtain $s_{a+b} \sum s_i = c_{a,b+2} + c_{a+1,b+1} - c_{a,b+1} + c_{a+2,b} - c_{a+1,b} - c_{a,b}$.

Now we know that $\sum s_i$ vanishes by (3.3.1) equation (3.4), and so

$$c_{a,b+2} + c_{a+1,b+1} - c_{a,b+1} + c_{a+2,b} - c_{a+1,b} - c_{a,b} = 0$$

as required.

These four systems of equations (3.8), (3.9), (3.10) and (3.13) will be crucial in the sequel. We shall give the systems special names; we call them *vertical, horizontal, diagonal* and *triangular* systems respectively. The reason for this should be clear if we think of the subscripts a, b as indexing integer points in a lattice. Gathering all the four systems under one heading, we shall refer to them as *template* equations.

Proposition 3.3.5 *Any collection of quantities satisfying the template equations must vanish. In particular we must have*

$$c_{a,b} = 0 \quad \forall\ a, b \in \mathbf{Z}. \tag{3.14}$$

We shall work with the specific quantities $c_{a.b}$, though the only properties of these quantities we shall use are the template equations. Thus the general version of (3.3.5) will be justified.

First we assume that $c_{a,b} = 0$ for all a and for some fixed b. We fix a reference point in the (a,b)−plane with an underscore. Let λ, μ, α and β be unknowns; then we have data

$$\begin{array}{cccc} \alpha & 2\alpha-\beta & * & * \\ \lambda & \alpha-\beta & * & * \\ \mu & \alpha & \beta & * \\ 0 & \underline{0} & 0 & 0 \end{array}.$$

The entries we have filled in are easy consequences of the template equations. We use $*$ to denote $c_{r,s}$ in the a, b−plane when we have not yet attempted to describe $c_{r,s}$. We now apply the triangular equation to the bottom left corner of our region to eliminate μ, and obtain

$$\begin{array}{cccc}
\alpha & 2\alpha-\beta & * & * \\
\lambda & \alpha-\beta & * & * \\
\lambda+\alpha & \alpha & \beta & * \\
0 & \underline{0} & 0 & 0
\end{array}.$$

Application of the vertical equation to the leftmost column of the diagram above yields that $\lambda = 0$. We update our diagram, and extend it a little both up and left using our template equations to yield

$$\begin{array}{ccccc}
2\alpha & 2\alpha & 4\alpha-2\beta & * & * \\
\alpha & \alpha & 2\alpha-\beta & * & * \\
3\alpha-\beta & 0 & \alpha-\beta & 4\alpha-2\beta & * \\
 & \alpha & \alpha & \beta & * \\
 & 0 & \underline{0} & 0 & 0
\end{array}.$$

In the diagram above, apply the triangle equation with the right angle placed on the 0 in the third row to obtain that $\beta = 3\alpha$. Insert this new information to give the table

$$\begin{array}{ccccc}
3\alpha & * & * & * & * \\
2\alpha & 2\alpha & -2\alpha & * & * \\
\alpha & \alpha & -\alpha & \alpha & 0 \\
0 & 0 & -2\alpha & -2\alpha & * \\
\alpha & \alpha & \alpha & 3\alpha & * \\
0 & 0 & \underline{0} & 0 & 0
\end{array}.$$

In this diagram, apply the diagonal equation to the leading diagonal to obtain that $\alpha = 0$. Now the recurrences force $c_{a,b} = 0$ for all values of a and b. from which it follows that $\alpha = 0$ and hence $c_{a,b} = 0$ for all integers a and b.

Next we assume that we have two, but not three, consecutive values of 0 in an 'a−row'. By linearity, it suffices to assume that the consecutive values are 0,0 and 1. We introduce new unknowns λ, μ, x and y. The equations show as that our patch of the a, b−plane can be described as

$$\begin{array}{cccc} \lambda & x-y-1 & * & * \\ \mu & x & y & * \\ 1 & \underline{0} & 0 & 1 \end{array}.$$

Apply the triangular equation in the top-leftmost position to find that $\lambda = \mu + 1 - x$. Our table is

$$\begin{array}{cccc} x & 2x-y-1 & * & * \\ \mu+1-x & x-y-1 & * & * \\ \mu & x & y & * \\ 1 & \underline{0} & 0 & 1 \end{array}.$$

We have filled in an extra fragment of a row on the top. Apply the triangular equation on the top-leftmost position to deduce that $\mu = x - 1$. Incorporate this new information, and extend the table to the left to reach

$$\begin{array}{ccccc} x & x & 2x-y-1 & * & * \\ y-3x+2 & 0 & x-y-1 & * & * \\ y-2x+1 & x-1 & x & y & 2x+y-1 \\ -1 & 1 & \underline{0} & 0 & 1 \end{array}$$

Once again we deploy the triangular equation in the top-leftmost position, and this time the dividend is that $y = 3x - 1$. Thus we may eliminate y from our table. Extend the diagram up and left by use of the diagonal equation to obtain

$$\begin{array}{cccccc} x & x & x & -x & * & * \\ & 1 & 0 & -2x & * & * \\ & x & x-1 & x & 3x-1 & 5x-2 \\ & -1 & 1 & \underline{0} & 0 & 1 \end{array}.$$

The horizontal equation applied to the top row yields that $x = 0$. Extending the table up in one position using the vertical equation, the table becomes

$$\begin{array}{cccccc} * & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 & -2 \\ 0 & -1 & 1 & \underline{0} & 0 & 1 \end{array}.$$

The diagonal equation applied to include the topmost 1 then yields 1=-1, a contradiction.

Now we suppose that we have an a-row containing just one consecutive 0, and so, by linearity, we may assume that our a-row contains $0, 1, x$ and that $x \neq 1$. We introduce unknowns in the usual way and can describe our patch of the a, b-plane as

$$\begin{array}{cccc} \alpha+1 & 2\alpha+1-\beta-x & * & * \\ \lambda & \alpha+1-\beta-x & * & * \\ \mu & \alpha & \beta & * \\ x-1 & 0 & 1 & x \end{array}.$$

The triangular equation applied in the bottom-left position yields that $\mu = \lambda+\alpha+2-x$, and so our table becomes

$$\begin{array}{cccc} \alpha+1 & 2\alpha+1-\beta-x & * & * \\ \lambda & \alpha+1-\beta-x & * & * \\ \lambda+\alpha+2-x & \alpha & \beta & * \\ x-1 & \underline{0} & 1 & x \end{array}.$$

Now apply the triangular equation in the top-leftmost position to obtain that $\lambda = 0$. We now extend the diagram to the left, and introduce two new unknowns. In the interests of conservation we recycle our Greek letters, and call these new unknowns λ and μ. We reach the position

$$\begin{array}{ccccc} \alpha+1 & \alpha+1 & 2\alpha+1-\beta-x & * & * \\ \lambda & 0 & \alpha+1-\beta-x & * & * \\ \mu & \alpha+2-x & \alpha & \beta & * \\ 2-x & x-1 & \underline{0} & 1 & x \end{array}.$$

Now deploy the triangular equation in each of the two possible leftmost positions of the diagram. We obtain two equations

$$\alpha - 1 + x = \mu + \lambda$$

and

$$\alpha + 1 - x = \mu - \lambda.$$

Solving these simultaneously we find that $\mu = \alpha$ and $\lambda = x - 1$. The fact that $\mu = \alpha$ forces $\beta = 3\alpha + 2 - x$ by the horizontal equation. We thus obtain

$$\begin{array}{ccccc} 2\alpha+x & * & * & * & * \\ \alpha+1 & \alpha+1 & -\alpha-1 & * & * \\ x-1 & 0 & -2\alpha-1 & * & * \\ \alpha & \alpha+2-x & \alpha & 3\alpha+2-x & * \\ 2-x & x-1 & \underline{0} & 1 & x \end{array}.$$

We have used the vertical equation to determine the entry $2\alpha + x$ in a new row. Now we use the diagonal equation as applies to the new entry to deduce that $x = 1$, and we are back in a case which we have eliminated previously, since $0, 0, 1$ appear as consecutive entries on the bottom row.

Finally we have only one remaining case to consider; that is the case when each $c_{a,b}$ is non-zero. In this instance, choose and fix an integer pair a', b' and, for all integers a and b, we put $C_{a,b} = c_{a,b} - c_{a',b'}$. By linearity the quantities $C_{a,b}$ satisfy the template equation. Also, $C_{a',b'} = 0$ so by the proof so far, we may deduce that $C_{a,b} = 0$ for all integers a and b. Thus $c_{a,b}$ must be constant (independent of a and b). Consider the horizontal equation (3.9); this forces $c_{a,b}$ to vanish for all a and b. This is absurd, and so we have finally established that

$$c_{a,b} = \sum s_i s_{i+a} s_{-i+b} = 0$$

for all $a, b \in \mathbf{Z}$, and the proof of (3.3.5) is complete.

Notice that it immediately follows that

Corollary 3.3.6 *For all integers α, β and γ we have*

$$\sum s_{i+\alpha} s_{i+\beta} s_{-i+\gamma} = 0.$$

Recall that we wish to show that $d_j = e_j = f_j = 0$ for $j \in \{k, k+1, k+2\}$. We begin with d_k. Recall that

$$d_k = \sum s_{k-i-1}(u_{i+1} t_i + u_{i+2}(t_i + t_{i+1})).$$

Now we eliminate the use of the sequences (u_i) and (t_i). We have $u_{i+1} = s_i$ and $t_i = s_{i+1} - s_i$. Thus we obtain

$$d_k = \sum s_{k-i-1} s_i (s_{i+1} - s_i) + \sum s_{k-i-1} s_{i+1}(s_{i+1} - s_i + s_{i+2} - s_{i+1}),$$

so that

$$d_k = \sum s_{k-i-1} s_i s_{i+1} - \sum s_{k-i-1} s_i^2 + \sum s_{k-i-1} s_{i+1} s_{i+2} - \sum s_{k-i-1} s_{i+1} s_i.$$

We apply (3.3.6) to each sum separately to obtain the result that $d_k = 0$.

Now for the other sums.

$$d_{k+1} = \sum_{i=0}^{k} s_{k-i}(u_{i+1} t_i + u_{i+2}(t_i + t_{i+1}));$$

but $s_0 = 0$, so that

$$d_{k+1} = \sum_{i=0}^{k-1} s_{k-i}(u_{i+1} t_i + u_{i+2}(t_i + t_{i+1}))$$

and

$$d_{k+1} = \sum s_{k-i}s_i(s_{i+1} - s_i) + \sum s_{k-i}s_{i+1}(s_{i+1} - s_i) + \sum s_{k-i}s_{i+1}(s_{i+2} - s_{i+1})$$

which vanish for the usual reasons. Now

$$d_{k+2} = \sum_{i=0}^{k+1} s_{k+1-i}(u_{i+1}t_i + u_{i+2}(t_i + t_{i+1}));$$

but we know that $s_0 = s_1 = 0$, and so we may discard k and $k+1$ from the range of i in this sum to yield

$$d_{k+2} = \sum_{i=0}^{k-1} s_{k+1-i}(u_{i+1}t_i + u_{i+2}(t_i + t_{i+1})).$$

Use the equations (3.1) and (3.2) to see that

$$d_{k+2} = \sum s_{k+1-i}s_i(s_{i+1} - s_i) + \sum s_{k+1-i}s_{i+1}(s_{i+1} - s_i) + \sum s_{k+1-i}s_{i+1}(s_{i+2} - s_{i+1}).$$

The sums on the right hand side vanish by (3.3.6).

We move on to discuss e_k. The argument is similar in character and so we spell out fewer details. We have

$$e_k = \sum s_{k-i-1}(s_iu_{i+1} + u_{i+2}(s_i + s_{i+1}))$$

so

$$e_k = \sum s_{k-i-1}s_i^2 + \sum s_{k-i-1}s_is_{i+1} + \sum s_{k-i-1}s_{i+1}^2.$$

Again we deploy (3.3.6) to each sum separately to obtain the result that $e_k = 0$.

Now

$$e_{k+1} = \sum_{i=0}^{k} s_{k-i}(s_iu_{i+1} + u_{i+2}(s_i + s_{i+1}));$$

since $s_0 = 0$, we have

$$e_{k+1} = \sum_{i=0}^{k-1} s_{k-i}(s_iu_{i+1} + u_{i+2}(s_i + s_{i+1}))$$

and so

$$e_{k+1} = \sum s_{k-i}s_i^2 + \sum s_{k-i}s_{i+1}s_i + \sum s_{k-i}s_{i+1}^2$$

and these sums vanish by (3.3.6). Similarly,

$$e_{k+2} = \sum_{i=0}^{k+1} s_{k-i+1}(s_i u_{i+1} + u_{i+2}(s_i + s_{i+1}));$$

but $s_0 = s_1 = 0$, and so we have

$$e_{k+2} = \sum_{i=0}^{k-1} s_{k-i+1}(s_i u_{i+1} + u_{i+2}(s_i + s_{i+1}));$$

then

$$e_{k+2} = \sum s_{k-i+1}s_i^2 + \sum s_{k-i+1}s_{i+1}s_i + \sum s_{k-i+1}s_{i+1}^2;$$

now apply (3.3.6) to each sum to get $e_{k+2} = 0$.

Next we tackle f_k. We have

$$f_k = \sum s_{k-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}));$$

so, by the familiar argument, f_k is a sum of terms of the form $c_{a,b}$. We deduce that $f_k = 0$.

Now

$$f_{k+1} = \sum_{i=0}^{k} s_{k-i}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}));$$

since $s_0 = 0$, we have

$$f_{k+1} = \sum s_{k-i}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}))$$

which is again a sum of terms of the form $c_{a,b}$; therefore $f_{k+1} = 0$.

Finally,

$$f_{k+2} = \sum_{i=0}^{k+1} s_{k-i+1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}));$$

but $s_0 = s_1 = 0$, so that

$$f_{k+2} = \sum s_{k-i+1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1})).$$

By a similar argument, $f_{k+2} = 0$. We have proved

Proposition 3.3.7 *Take any three free generators x, y, z of the 3-generator relatively free group G in the variety of nilpotent groups of class 2 and exponent p. Form a 3-step Fibonacci sequence using these generators as initial data. The fundamental period of this sequence is $k(p)$.*

Any 3−step Fibonacci sequence in any finite p−group P of exponent p lives in a 3-generator subgroup H of P, and there is a homomorphism from the free group to this group H sending initial data to initial data. Thus (3.3.7) immediately gives us

Theorem 3.3.8 *If P is a finite p−group of exponent $p > 3$ and nilpotency class 2 then $k(P)$ divides k.*

In fact we can replace '$k(P)$ divides k' with $k(P) = k$ unless P is the trivial group. This is because $k(C_p) = k$.

3.4 Proof: Nilpotency Class 3

We now move on to study the situation where the nilpotency class of the groups rises to 3. We are still interested in the 3−step Fibonacci recurrence. We do our preliminary investigations, not with the relatively free group on three generators, but with a carefully selected group H which we now describe. H has two generators x and y. A presentation of H is

$$H = < h_1, h_2, h_3, h_4 \ : \ (h_2, h_1) = h_3, \ (h_3, h_1) = h_4, \ explaw = p \ >$$

where pairs of generators with unspecified commutator are implicitly deemed to commute. Thus H is a copy of C_p^3 extended by a cyclic group of order p.

Let G be the 3−generator relatively free exponent p class 3 group on g_1, g_2 and g_3. Thus G has order p^{14} and a power commutator presentation of G is given by

$$\begin{aligned}
(g_2, g_1) &= g_4 \\
(g_3, g_1) &= g_5 \\
(g_3, g_2) &= g_6 \\
(g_4, g_1) &= g_7 \\
(g_4, g_2) &= g_8 \\
(g_4, g_3) &= g_9 \\
(g_5, g_1) &= g_{10} \\
(g_5, g_2) &= g_{11} \\
(g_5, g_3) &= g_{12} \\
(g_6, g_1) &= g_9^{-1} g_{11} \\
(g_6, g_2) &= g_{13} \\
(g_6, g_3) &= g_{14}
\end{aligned}$$

Once again we have the convention that pairs of generators with unspecified commutator are implicitly deemed to commute.

In $GF(p)$−vector notation, we put $g_i = (\delta_{ij}) \in G$, where δ_{ij} in the Krönecker symbol, and j ranges from 1 to 14.

The group G is relatively free and so admits an automorphism ϕ, which we call the 3−step *Fibonacci automorphism*, defined by $g_1\phi = g_2,\ g_2\phi = g_3$ and $g_3\phi = g_1 g_2 g_3$.

We define two maps $\pi_i : G \longrightarrow H$ via

$$g_1\pi_1 = 1,\ g_2\pi_1 = h_1 \text{ and } g_3\pi_1 = h_2$$

and

$$g_1\pi_2 = h_1,\ g_2\pi_2 = h_2 \text{ and } g_3\pi_2 = 1.$$

Let $g_i = (\delta_{ij}) \in G$, where δ_{ij} in the Krönecker symbol, and j ranges from 1 to 14.

$$Ker\pi_1 = K_1 = (*,0,0,*,*,0,*,*,*,*,*,*,0,*);$$

$$Ker\pi_2 = K_2 = (0,0,*,0,*,*,0,*,*,*,*,*,*,*).$$

Let $M = Ker\pi_1 \cap Ker\pi_2$, so that

$$M = (0,0,0,0,*,0,0,*,*,*,*,*,0,*),$$

in the sense that each $*$ can independently be any element of $GF(p)$. Now M is an elementary abelian group of order p^7, and is therefore a $GF(p)$-space of dimension 7. A basis of M is $(g_5, g_8, g_9, g_{10}, g_{11}, g_{12}, g_{14})$.

Computer aided calculations [13] yield that

$$M \cap M\phi = (g_8 g_{11}, g_9, g_{10} g_{11} g_{12}, g_{14}),$$

$$M \cap M\phi \cap M\phi^2 = (g_9 g_{14}^{-2}, g_{10} g_{11} g_{12})$$

and

$$M \cap M\phi \cap M\phi^2 \cap M\phi^3 = 1.$$

Thus we have a monomorphism

$$\pi : G \longrightarrow G/K_1 \times G/K_2 \times G/K_1\phi \times G/K_2\phi \times G/K_1\phi^2 \times G/K_2\phi^2 \times G/K_1\phi^3 \times G/K_2\phi^3,$$

where the codomain is isomorphic to $\times_{i=1}^{8} H$. The automorphism ϕ^{-1} and its powers induce isomorphisms $G/K_i\phi^j \longrightarrow G/K_i$ which can be composed co-ordinatewise with π to form a group monomorphism

$$\bar{\pi} : G \longrightarrow \times_{j=1}^{4}(G/K_1 \times G/K_2)$$

defined by

$$x \longrightarrow (K_1x, K_2x, K_1(x\phi^{-1}), K_2(x\phi^{-1}), K_1(x\phi^{-2}), K_2(x\phi^{-2}), K_1(x\phi^{-3}), K_2(x\phi^{-3})).$$

Now let us examine the image of the loop $r = (r_i)$ beginning $r_0 = g_1$, $r_1 = g_2$, $r_2 = g_3$ under $\bar{\pi}$. We have

$$\bar{\pi} : r_i \longrightarrow (r_i\pi_1, r_i\pi_2, r_{i-1}\pi_1, r_{i-1}\pi_2, r_{i-2}\pi_1, r_{i-2}\pi_2, r_{i-3}\pi_1, r_{i-3}\pi_2).$$

The sequences in the odd positions are just rotations of $(r_i\pi_1)$ and the sequences in the even positions are rotations of $(r_i\pi_2)$. Thus, if we can show that $(r_i\pi_1)$ and $(r_i\pi_2)$ both have Wall Number k, it will follow that r has Wall Number k and we will be done.

In H the elements can be regarded as vectors in the usual way, and triple multiplication is determined by the following rules;

$$(a_0, b_0, c_0, d_0) \cdot (a_1, b_1, c_1, d_1) \cdot (a_2, b_2, c_2, d_2) = (a_3, b_3, c_3, d_3)$$

where

$$a_3 = a_0 + a_1 + a_2,$$

$$b_3 = b_0 + b_1 + b_2,$$

$$c_3 = c_0 + c_1 + c_2 + a_1b_0 + a_2(b_0 + b_1),$$

and finally

$$d_3 = d_0 + d_1 + d_2 + a_1c_0 + a_2(c_0 + c_1 + a_1b_0) + \tbinom{a_2}{2}(b_0 + b_1) + \tbinom{a_1}{2}b_0.$$

Notice that these sequences and multiplication rules are, of course, different from those in the previous section.

We must consider two types of initial data for loops in H. We have a loop v of type

I with initial data

$$\begin{aligned} v_0 &= (0,0,0,0) \\ v_1 &= (1,0,0,0) \\ v_2 &= (0,1,0,0) \end{aligned}$$

and another, w, of type II with initial data

$$\begin{aligned} w_0 &= (1,0,0,0) \\ w_1 &= (0,1,0,0) \\ w_2 &= (0,0,0,0). \end{aligned}$$

The analysis of the type II loop is entirely similar to that of type I. Thus the type I loop begins

$$\begin{aligned} v_0 &= (t_0, s_0, 0, 0) \\ v_1 &= (t_1, s_1, 0, 0) \\ v_2 &= (t_2, s_2, 0, 0). \end{aligned}$$

We focus on the type I loop $(v_i) = (t_i, s_i, c_i, d_i)$. Now, it follows immediately from (3.3.8) that $c_k = c_{k+1} = c_{k+2} = 0$. We must demonstrate that $d_k = d_{k+1} = d_{k+2} = 0$. We shall first show that $d_k = 0$.

We shall need a formula for c_α in order to work out the formula for d_α. By induction it is

$$c_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}))$$

for $\alpha \geq 0$. This enables us, via a similar process, to describe d_α for $\alpha \geq 0$ as

$$d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+1} c_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+1}}{2} s_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+2}(c_i + c_{i+1} + t_{i+1} s_i)$$

$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2} (s_i + s_{i+1}).$$

We can break up the expression for d_k as $d_k = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ where

$$\Delta_1 = \sum s_{k-i-1} t_{i+1} c_i$$

$$\Delta_2 = \sum s_{k-i-1}\tbinom{t_{i+1}}{2} s_i$$

$$\Delta_3 = \sum s_{k-i-1}t_{i+2}(c_i + c_{i+1} + t_{i+1}s_i)$$

and

$$\Delta_4 = \sum s_{k-i-1}\tbinom{t_{i+2}}{2}(s_i + s_{i+1}),$$

and we shall attempt to show that each of these four expressions Δ_i actually vanishes. To this end, we break these expressions up still further.

Now we have

$$\Delta_1 = \sum s_{k-i-1}t_{i+1}c_i = \sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}c_j$$

$$= \sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}\Big(\sum_{i=0}^{j-1} s_{j-i-1}s_i t_{i+1} + \sum_{i=0}^{j-1} s_{j-i-1}t_{i+2}(s_i + s_{i+1})\Big),$$

and so $\Delta_1 = \Delta_{11} + \Delta_{12} + \Delta_{13}$, where

$$\Delta_{11} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_i t_{i+1},$$

$$\Delta_{12} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_i t_{i+2}$$

and

$$\Delta_{13} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+1}t_{i+2}.$$

Moving to Δ_2, we find that

$$\Delta_2 = \sum_{j=0}^{k-1} s_{k-j-1}\tbinom{t_{j+1}}{2}s_j = \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}(t_{j+1} - 1)s_j,$$

so that

$$\Delta_2 = \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}^2 s_j - \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}s_j.$$

Next we tackle Δ_3. We have

$$\Delta_3 = \sum_{j=0}^{k-1} s_{k-j-1}t_{j+2}(c_j + c_{j+1} + t_{j+1}s_j),$$

so that

$$\Delta_3 = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (\sum_{i=0}^{j-1} s_{j-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1})))$$

$$+ \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (\sum_{i=0}^{j-1} s_{j-i}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1})))$$

$$+ \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} t_{j+1} s_j .$$

Thus $\Delta_3 = \Delta_{31} + \Delta_{32} + \Delta_{33} + \Delta_{34} + \Delta_{35} + \Delta_{36} + \Delta_{37}$ where

$$\Delta_{31} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_i t_{i+1},$$

$$\Delta_{32} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_i t_{i+2},$$

$$\Delta_{33} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_{i+1} t_{i+2},$$

$$\Delta_{34} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i} s_i t_{i+1},$$

$$\Delta_{35} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i} s_i t_{i+2},$$

$$\Delta_{36} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i} s_{i+1} t_{i+2}$$

and

$$\Delta_{37} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} t_{j+1} s_j .$$

The notation $\sum_{i<j}$ indicates that we are dealing with a double sum, taken over all i and j subject to the constraint that $0 \le i < j \le k-1$. Also we see that

$$\Delta_4 = \sum_{j=0}^{k-1} s_{k-j-1} \binom{t_{j+2}}{2} (s_j + s_{j+1}),$$

so that

$$\Delta_4 = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (t_{j+2} - 1)(s_j + s_{j+1});$$

but $\Delta_4 = \Delta_{41} - \Delta_{42}$ where

$$\Delta_{41} = \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1} t_{j+2}^2 (s_j + s_{j+1})$$

and

$$\Delta_{42} = \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (s_j + s_{j+1}).$$

We want to show all sums of type Δ actually vanish. To this end, we prove some lemmas.

Lemma 3.4.1 *For all integers α and β we have*

$$\sum_{i<j} s_{j+\alpha} s_{i+\beta} = 0. \tag{3.15}$$

Proof We have

$$\sum_{i<j} s_{j+\alpha} s_{i+\beta} = \sum_{i<j} s_{j+\alpha}(s_{i+1+\beta} - s_{i-1+\beta} - s_{i-2+\beta}),$$

so that

$$\sum_{i<j} s_{j+\alpha} s_{i+\beta} = \sum_{i<j} s_{j+\alpha} s_{i+1+\beta} - \sum_{i<j} s_{j+\alpha} s_{i-1+\beta} - \sum_{i<j} s_{j+\alpha} s_{i-2+\beta}.$$

We manipulate the three sums on the right hand side of this equation. The first sum is

$$\sum_{i<j} s_{j+\alpha} s_{i+1+\beta} = \sum_{j=0}^{k-1}\sum_{i=1}^{j} s_{j+\alpha} s_{i+\beta} = \sum_{i<j} s_{j+\alpha} s_{i+\beta} - \sum_{j=0}^{k-1} s_{j+\alpha} s_{\beta} + \sum_{j=0}^{k-1} s_{j+\alpha} s_{j+\beta}.$$

The last two sums vanish by equations (3.4) and (3.7) respectively.

We now address the second sum. We have

$$\sum_{i<j} s_{j+\alpha} s_{i-1+\beta} = \sum_{j=0}^{k-1}\sum_{i=-1}^{j-2} s_{j+\alpha} s_{i+\beta},$$

so that

$$\sum_{i<j} s_{j+\alpha} s_{i-1+\beta} = \sum_{i<j} s_{j+\alpha} s_{i+\beta} + \sum_{j=0}^{k-1} s_{j+\alpha} s_{-1+\beta} - \sum_{j=0}^{k-1} s_{j+\alpha} s_{j-1+\beta}.$$

The last two sums vanish, by equations (3.4) and (3.7).

Finally we address the third sum. We have

$$\sum_{i<j} s_{j+\alpha}s_{i-2+\beta} = \sum_{j=0}^{k-1}\sum_{i=-2}^{j-3} s_{j+\alpha}s_{i+\beta},$$

so that

$$\sum_{i<j} s_{j+\alpha}s_{i-2+\beta} = \sum_{i<j} s_{j+\alpha}s_{i+\beta} + \sum_{j=0}^{k-1} s_{j+\alpha}s_{-2+\beta} + \sum_{j=0}^{k-1} s_{j+\alpha}s_{-1+\beta}$$
$$-\sum_{j=0}^{k-1} s_{j+\alpha}s_{j-2+\beta} - \sum_{j=0}^{k-1} s_{j+\alpha}s_{j-1+\beta}.$$

The last four sums vanish by deploying (3.4) and (3.7) again.

Thus we have

$$\sum_{i<j} s_{j+\alpha}s_{i+\beta} = \sum_{i<j} s_{j+\alpha}s_{i+\beta} - \sum_{i<j} s_{j+\alpha}s_{i+\beta} - \sum_{i<j} s_{j+\alpha}s_{i+\beta}.$$

Thus

$$\sum_{i<j} s_{j+\alpha}s_{i+\beta} = -\sum_{i<j} s_{j+\alpha}s_{i+\beta},$$

and so we are done.

Now we prove the second lemma on this theme.

Lemma 3.4.2 *For all integers α, β and c, we have*

$$\sum_{i<j} s_{j-i+\alpha}s_{i+c}s_{i+\beta} = 0. \tag{3.16}$$

Proof We fix c (which is why it is in a different alphabet) and put

$$d_{\alpha,\beta} = \sum_{i<j} s_{j-i+\alpha}s_{i+c}s_{i+\beta}.$$

We claim that the following systems of equations are all satisfied.

$$d_{\alpha,\beta} = d_{\alpha-1,\beta} + d_{\alpha-2,\beta} + d_{\alpha-3,\beta},$$

$$d_{\alpha,\beta} = d_{\alpha,\beta-1} + d_{\alpha,\beta-2} + d_{\alpha,\beta-3},$$

$$d_{\alpha,\beta} = d_{\alpha-1,\beta+1} + d_{\alpha-2,\beta+2} + d_{\alpha-3,\beta+3}$$

and

$$d_{\alpha,\beta+2} + d_{\alpha+1,\beta+1} - d_{\alpha,\beta+1} + d_{\alpha+2,\beta} - d_{\alpha+1,\beta} - d_{\alpha,\beta} = 0.$$

The first, second and fourth systems are all template equations. and the third is almost the diagonal system – but the 'direction' of the equation is wrong.

The first two systems of equations are easy consequences of the Fibonacci recurrence. The third equation is a little more complex. We have

$$d_{\alpha,\beta} = \sum_{i<j} s_{j-i+\alpha}(s_{i-1+c} + s_{i-2+c} + s_{i-3+c})s_{i+\beta},$$

so that

$$d_{\alpha,\beta} = \sum_{i<j} s_{j-i+\alpha}s_{i-1+c}s_{i+\beta} + \sum_{i<j} s_{j-i+\alpha}s_{i-2+c}s_{i+\beta} + \sum_{i<j} s_{j-i+\alpha}s_{i-3+c}s_{i+\beta}. \tag{3.17}$$

We examine the three sums on the right of (3.17) separately. The first one is

$$\sum_{i<j} s_{j-i+\alpha}s_{i-1+c}s_{i+\beta} = \sum_{j=0}^{k-1}\sum_{i=-1}^{j-2} s_{j-i-1+\alpha}s_{i+c}s_{i+1+\beta},$$

so that

$$\sum_{i<j} s_{j-i+\alpha}s_{i-1+c}s_{i+\beta} = \sum_{i<j} s_{j-i-1+\alpha}s_{i+c}s_{i+1+\beta} + \sum_{j=0}^{k-1} s_{j+\alpha}s_{-1+c}s_{\beta} - \sum_{j=0}^{k-1} s_{\alpha}s_{j-1+c}s_{j+\beta}.$$

The second sum and third sums on the right of this equation vanish by (3.4) and (3.7) respectively. Thus the first sum on the right of equation (3.17) is just $d_{\alpha-1,\beta+1}$. Now the second is

$$\sum_{i<j} s_{j-i+\alpha}s_{i-2+c}s_{i+\beta} = \sum_{j=0}^{k-1}\sum_{i=-2}^{j-3} s_{j-i-2+\alpha}s_{i+c}s_{i+2+\beta},$$

so that

$$\sum_{i<j} s_{j-i+\alpha}s_{i-2+c}s_{i+\beta} = \sum_{i<j} s_{j-i-2+\alpha}s_{i+c}s_{i+2+\beta} + \sum_{j=0}^{k-1} s_{j+\alpha}s_{-2+c}s_{\beta} + \sum_{j=0}^{k-1} s_{j-1+\alpha}s_{-1+c}s_{\beta+1}$$

$$-\sum_{j=0}^{k-1} s_{\alpha}s_{j-2+c}s_{j+\beta} - \sum_{j=0}^{k-1} s_{\alpha-1}s_{j-1+c}s_{j+1+\beta}.$$

The two sums on the right of this equation vanish by (3.4) and the last two by (3.7). Therefore the second sum on the right of equation (3.17) is exactly $d_{\alpha-2,\beta+2}$. Finally, the third sum is

$$\sum_{i<j} s_{j-i+\alpha}s_{i-3+c}s_{i+\beta} = \sum_{j=0}^{k-1}\sum_{i=-3}^{j-4} s_{j-i-3+\alpha}s_{i+c}s_{i+3+\beta};$$

then

$$\sum_{i<j} s_{j-i+\alpha}s_{i-3+c}s_{i+\beta} = \sum_{i<j} s_{j-i-3+\alpha}s_{i+c}s_{i+3+\beta} + \sum_{j=0}^{k-1} s_{j+\alpha}s_{-3+c}s_{\beta}$$

$$+\sum_{j=0}^{k-1} s_{j-1+\alpha}s_{-2+c}s_{\beta+1} + \sum_{j=0}^{k-1} s_{j-2+\alpha}s_{-1+c}s_{\beta+2} - \sum_{j=0}^{k-1} s_{\alpha}s_{j-3+c}s_{j+\beta}$$

$$-\sum_{j=0}^{k-1} s_{\alpha-1}s_{j-2+c}s_{j+1+\beta} - \sum_{j=0}^{k-1} s_{\alpha-2}s_{j-1+c}s_{j+2+\beta}.$$

All the single sums on the right of this equation vanish by (3.4) and (3.7). Thus the last sum on the right of equation (3.17) is $d_{\alpha-3,\beta+3}$ and our equation is established.

We now tackle the fourth system of equations. Recall that we have established the following equation (3.11). For all integers α and β we have

$$s_{\alpha+\beta} = s_{\alpha}s_{\beta+2} + (s_{\alpha+1} - s_{\alpha})s_{\beta+1} + (s_{\alpha+2} - s_{\alpha+1} - s_{\alpha})s_{\beta}.$$

Replace α by $j - i + \alpha$ and β by $i + \beta$ to obtain

$$s_{j+\alpha+\beta} = s_{j-i+\alpha}s_{i+\beta+2} + (s_{j-i+\alpha+1} - s_{j-i+\alpha})s_{i+\beta+1} + (s_{j-i+\alpha+2} - s_{j-i+\alpha+1} - s_{j-i+\alpha})s_{i+\beta}.$$

Now multiply this equation by s_{i+c}, and then sum over the range $0 \le i < j < k$, to obtain

$$\sum_{i<j} s_{j+\alpha+\beta} s_{i+c} = d_{\alpha,\beta+2} + d_{\alpha+1,\beta+1} - d_{\alpha,\beta+1} + d_{\alpha+2,\beta} - d_{\alpha+1,\beta} - d_{\alpha,\beta}.$$

We now deploy equation (3.15) of (3.4.1) to deduce that

$$d_{\alpha,\beta+2} + d_{\alpha+1,\beta+1} - d_{\alpha,\beta+1} + d_{\alpha+2,\beta} - d_{\alpha+1,\beta} - d_{\alpha,\beta} = 0.$$

Thus the 'unknowns' $d_{a,b}$ satisfy the same system of equations as the unknowns $c_{b,a}$ – notice the transposition of subscripts. Thus each $d_{a,b}$ vanishes by (3.3.5) and so we are done.

Lemma 3.4.3 *For all $\alpha, \beta, \gamma \in \mathbf{Z}$ we have*

$$\sum_{j=0}^{k-1} s_{j+\alpha} s_{j+\beta} s_{-j+\gamma} s_j = 0. \tag{3.18}$$

Proof Let

$$X_{\alpha,\beta,\gamma} = \sum_{j=0}^{k-1} s_{j+\alpha} s_{j+\beta} s_{-j+\gamma} s_j,$$

and consider the sequence

$$Y_{\alpha,\beta} = (X_{\alpha,\beta,0}, X_{\alpha,\beta,1}, X_{\alpha,\beta,2}, \ldots)$$

and its rotations

$$Y^+_{\alpha,\beta} = (X_{\alpha,\beta,1}, X_{\alpha,\beta,2}, X_{\alpha,\beta,3}, \ldots)$$

and

$$Y^-_{\alpha,\beta} = (X_{\alpha,\beta,-1}, X_{\alpha,\beta,0}, X_{\alpha,\beta,1}, \ldots).$$

We have the following equations as trivial consequences of the 3–step Fibonacci equation.

$$X_{\alpha,\beta,\gamma} = X_{\alpha,\beta-1,\gamma} + X_{\alpha,\beta-2,\gamma} + X_{\alpha,\beta-3,\gamma},$$

$$X_{\alpha,\beta,\gamma} = X_{\alpha-1,\beta,\gamma} + X_{\alpha-2,\beta,\gamma} + X_{\alpha-3,\beta,\gamma},$$

$$X_{\alpha,\beta,\gamma} = X_{\alpha,\beta,\gamma-1} + X_{\alpha,\beta,\gamma-2} + X_{\alpha,\beta,\gamma-3}$$

and

$$X_{\alpha,\beta,\gamma} = X_{\alpha+1,\beta+1,\gamma-1} + X_{\alpha+2,\beta+2,\gamma-2} + X_{\alpha+3,\beta+3,\gamma-3}.$$

There is also the less obvious equation of triangular type

$$0 = X_{\alpha,\beta,\gamma} + (X_{\alpha+1,\beta,\gamma-1} - X_{\alpha,\beta,\gamma-1}) + (X_{\alpha+2,\beta,\gamma-2} - X_{\alpha+1,\beta,\gamma-2} - X_{\alpha,\beta,\gamma-2}),$$

which follows from a version of (3.11)

$$s_{\alpha+\gamma} = s_{j+\alpha}s_{-j+\gamma} + (s_{j+\alpha+1} - s_{j+\alpha})s_{-j+\gamma-1} + (s_{j+\alpha+2} - s_{j+\alpha+1} - s_{j+\alpha})s_{-j+\gamma-2}$$

by multiplying by $s_j s_{j+\beta}$ and summing j over a fundamental range.

We may take these equations ($X-$equations) and turn them into equations among the quantities $Y_{a,b}$. We obtain a list of systems of equations holding for all integers a and b :

$$\begin{aligned}
Y_{a,b} &= Y_{a,b-1} + Y_{a,b-2} + Y_{a,b-3}, & (3.19)\\
Y_{a,b} &= Y_{a-1,b} + Y_{a-2,b} + Y_{a-3,b}, & (3.20)\\
Y_{a,b}^{+++} &= Y_{a,b}^{++} + Y_{a,b}^{+} + Y_{a,b}, & (3.21)\\
Y_{a,b}^{+++} &= Y_{a+1,b+1}^{++} + Y_{a+2,b+2}^{+} + Y_{a+3,b+3} & (3.22)\\
and Y_{a+2,b} &= -Y_{a,b}^{++} - Y_{a+1,b}^{+} + Y_{a,b}^{+} + Y_{a+1,b} + Y_{a,b}. & (3.23)
\end{aligned}$$

If we represent each $Y_{a,b}$ as a 3-entry column vector $(X_{a,b,0}, X_{a,b,1}, X_{a,b,2})^T$, then rotation is accomplished by left multiplication by the matrix

$$c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The equation system (3.23) can be reformulated as

$$Y_{\alpha+2,\beta} = (1 + c - c^2)Y_{\alpha,\beta} + (1 - c)Y_{\alpha+1,\beta},$$

where each Y is interpreted as a column vector. Let $A = 1 + c - c^2$ and $B = 1 - c$, so that we can recast our latest equation system as

$$Y_{\alpha+2,\beta} = AY_{\alpha,\beta} + BY_{\alpha+1,\beta}, \tag{3.24}$$

an equation system of horizontal type. Since the definition of $Y_{a,b}$ is symmetric in a and b, we have a similar system of vertical equations

$$Y_{\alpha,\beta+2} = AY_{\alpha,\beta} + BY_{\alpha,\beta+1}. \tag{3.25}$$

Put

$$\bar{a} = Y_{\alpha,\beta} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)^T,$$

$$\bar{b} = Y_{\alpha,\beta+1} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)^T,$$

$$\bar{c} = Y_{\alpha+1,\beta} = (\bar{c}_1, \bar{c}_2, \bar{c}_3)^T$$

and

$$\bar{d} = Y_{\alpha+1,\beta+1} = (\bar{d}_1, \bar{d}_2, \bar{d}_3)^T$$

and consider the 5 by 5 matrix:

$$\begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ A\bar{a} + B\bar{c} & A\bar{b} + B\bar{d} & * & * & * \\ \bar{c} & \bar{d} & A\bar{c} + B\bar{d} & * & * \\ \bar{a} & \bar{b} & A\bar{a} + B\bar{b} & A\bar{b} + BA\bar{a} + B^2\bar{b} & * \end{pmatrix}.$$

We use (3.24) and (3.25) to complete this 5 by 5 array. We can then use the earlier Y-equations to write down 12 simultaneous linear equations among the 12 unknowns

$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$. The determinant of the matrix of these equations was evaluated on a computer using the Computer Algebra system AXIOM [35] and found to be $2^{14}3^3$. Thus, save for the primes 2 and 3, we may deduce that $X_{\alpha,\beta,\gamma} = 0$ for all integers α, β and γ.

Lemma 3.4.4 *For all integers α, β, c, d and e we have*

$$\sum_{i<j} s_{-j+\alpha} s_{j+\beta} s_{j-i-d} s_{i+e} s_{i+c} = 0. \tag{3.26}$$

Proof Fix c, d and e for the purposes of the proof. Let

$$\theta_{\alpha,\beta} = \sum s_{-j+\alpha} s_{j+\beta} s_{j-i-d} s_{i+e} s_{i+c}$$

for all α, β and fixed c, d and e. Replace $s_{-j+\alpha}$ by $s_{-j+\alpha-1} + s_{-j+\alpha-2} + s_{-j+\alpha-3}$ to obtain

$$\theta_{\alpha,\beta} = \theta_{\alpha-1,\beta} + \theta_{\alpha-2,\beta} + \theta_{\alpha-3,\beta}.$$

Similarly we obtain

$$\theta_{\alpha,\beta} = \theta_{\alpha,\beta-1} + \theta_{\alpha,\beta-2} + \theta_{\alpha,\beta-3}.$$

Using (3.11) in the usual way we have

$$s_{\alpha+\beta} = s_{-j+\alpha} s_{\beta+j+2} + (s_{-j+\alpha+1} - s_{-j+\alpha}) s_{\beta+j+1} + (s_{-j+\alpha+2} - s_{-j+\alpha+1} - s_{-j+\alpha}) s_{\beta+j}.$$

Multiply both sides of this equation by $s_{j-i-d} s_{i+e} s_{i+c}$ and take a double sum over the fundamental range, i.e. over i and j such that $0 \leq i < j \leq k-1$. We obtain

$$\sum_{i<j} s_{\alpha+\beta} s_{j-i-d} s_{i+e} s_{i+c} = \theta_{\alpha,\beta+2} + \theta_{\alpha+1,\beta+1} - \theta_{\alpha,\beta+1} + \theta_{\alpha+2,\beta} - \theta_{\alpha+1,\beta} - \theta_{\alpha,\beta}.$$

The left hand side vanishes by (3.4.2) equation (3.16) and so a triangular equation is satisfied:

$$\theta_{\alpha,\beta+2} + \theta_{\alpha+1,\beta+1} - \theta_{\alpha,\beta+1} + \theta_{\alpha+2,\beta} - \theta_{\alpha+1,\beta} - \theta_{\alpha,\beta} = 0.$$

One can do an extended calculation of the usual kind to show that

$$\theta_{\alpha,\beta} = \theta_{\alpha+1,\beta-1} + \theta_{\alpha+2,\beta-2} + \theta_{\alpha+3,\beta-3},$$

a diagonal equation. The details are as follows.

Using the recurrence we know that

$$\theta_{\alpha,\beta} = \sum_{i<j} s_{-j+\alpha} s_{j+\beta} (s_{j-i-d-1} + s_{j-i-d-2} + s_{j-i-d-3}) s_{i+e} s_{i+c},$$

so that

$$\begin{aligned}\theta_{\alpha,\beta} = \textstyle\sum_{i<j} s_{-j+\alpha} s_{j+\beta} s_{j-i-d-1} s_{i+e} s_{i+c} + \sum_{i<j} s_{-j+\alpha} s_{j+\beta} s_{j-i-d-2} s_{i+e} s_{i+c} \\ + \textstyle\sum_{i<j} s_{-j+\alpha} s_{j+\beta} s_{j-i-d-3} s_{i+e} s_{i+c}.\end{aligned} \tag{3.27}$$

We address the first sum on the right of (3.27). We have

$$\sum_{i<j} s_{-j+\alpha} s_{j+\beta} (s_{j-i-d-1}) s_{i+e} s_{i+c} = \sum_{j=-1}^{k-2} \sum_{i=0}^{j} s_{-j+\alpha-1} s_{j+\beta+1} s_{j-i-d} s_{i+e} s_{i+c},$$

which in turn we can break up into

$$\begin{aligned}\textstyle\sum_{i<j} s_{-j+\alpha-1} s_{j+\beta+1} s_{j-i-d} s_{i+e} s_{i+c} - \sum s_\alpha s_\beta s_{k-i-d-1} s_{i+e} s_{i+c} \\ + \textstyle\sum s_{-j+\alpha-1} s_{j+\beta+1} s_{-d} s_{j+e} s_{j+c}.\end{aligned} \tag{3.28}$$

The second sum in expression (3.28) vanishes by (3.3.6) and the third, after a translation of variable, by (3.4.3).

Thus the first sum on the right of (3.27) is really just $\theta_{\alpha-1,\beta+1}$. Now consider the second sum on the right of (3.27). We have

$$\sum_{i<j} s_{-j+\alpha} s_{j+\beta} (s_{j-i-d-2}) s_{i+e} s_{i+c} = \sum_{j=-2}^{k-3} \sum_{i=0}^{j+1} s_{-j+\alpha-2} s_{j+\beta+2} s_{j-i-d} s_{i+e} s_{i+c},$$

so that

$$\sum_{i<j} s_{-j+\alpha}s_{j+\beta}(s_{j-i-d-2})s_{i+e}s_{i+c} = \sum_{i<j} s_{-j+\alpha-2}s_{j+\beta+2}s_{j-i-d}s_{i+e}s_{i+c}$$

$$-\sum s_\alpha s_\beta s_{k-i-d-2}s_{i+e}s_{i+c} - \sum s_{\alpha-1}s_{\beta+1}s_{k-i-d-1}s_{i+e}s_{i+c}$$

$$+\sum s_{-j+\alpha+2}s_{j+\beta+2}s_{-d}s_{j+e}s_{j+c} + \sum s_{-j+\alpha+2}s_{j+\beta+2}s_{-d-1}s_{j+e+1}s_{j+c+1}.$$

The first two single sums on the right of this equation vanish by (3.3.6) and last two by (3.4.3). Therefore the second sum on the right of (3.27) is $\theta_{\alpha-2,\beta+2}$. Finally, we consider the third sum on the right of (3.27). We have

$$\sum_{i<j} s_{-j+\alpha}s_{j+\beta}(s_{j-i-d-3})s_{i+e}s_{i+c} = \sum_{j=-3}^{k-4}\sum_{i=0}^{j+2} s_{-j+\alpha-3}s_{j+\beta+3}s_{j-i-d}s_{i+e}s_{i+c}$$

which in turn we break up into

$$-\sum s_\alpha s_\beta s_{k-i-d-3}s_{i+e}s_{i+c} - \sum s_{\alpha-1}s_{\beta+1}s_{k-i-d-2}s_{i+e}s_{i+c}$$

$$-\sum s_{\alpha-2}s_{\beta+2}s_{k-i-d-1}s_{i+e}s_{i+c} + \sum s_{-j+\alpha-3}s_{j+\beta+3}s_{-d}s_{j+e}s_{j+c}$$

$$+\sum s_{-j+\alpha-3}s_{j+\beta+3}s_{-d-1}s_{j+e+1}s_{j+c+1} + \sum s_{-j+\alpha-3}s_{j+\beta+3}s_{-d-2}s_{j+e+2}s_{j+c+2}.$$

The first three sums vanish by (3.3.6) and last three by (3.4.3). This means the third sum of (3.27) is actually $\theta_{\alpha-3,\beta+3}$. Thus we have established the diagonal equation:

$$\theta_{\alpha,\beta} = \theta_{\alpha+1,\beta-1} + \theta_{\alpha+2,\beta-2} + \theta_{\alpha+3,\beta-3}.$$

Now we see that the system of unknowns $\theta_{a,b}$ satisfy exactly the same template equations as the system $d_{a,b}$. Thus, by (3.3.5), we may deduce that $\theta_{\alpha,\beta} = 0$ for all integers α and β and (3.26) is established.

Now finally we can close in on the proof of (3.2.1). We need to show that all the Δ expressions vanish. We have the means at our disposal to do this. In the equations which follow, recall that we may replace any t_α by $s_{\alpha+1} - s_\alpha$ thanks to equation (3.1).

Recall that $\Delta_1 = \Delta_{11} + \Delta_{12} + \Delta_{13}$ where

$$\Delta_{11} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_i t_{i+1},$$

$$\Delta_{12} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_i t_{i+2}$$

and

$$\Delta_{13} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+1}t_{i+2}.$$

Now we have $\Delta_{11} = \Delta_{12} = \Delta_{13} = 0$ by (3.4.4).

Moving on to Δ_2, we recall that

$$\Delta_2 = \sum_{j=0}^{k-1} s_{k-j-1}\binom{t_{j+1}}{2}s_j = \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}(t_{j+1}-1)s_j,$$

so that

$$\Delta_2 = \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}^2 s_j - \frac{1}{2}\sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}s_j.$$

The first of these two sums vanishes by (3.4.3) and the second by (3.3.6). Thus we have $\Delta_2 = 0$.

Now we address Δ_3. We have $\Delta_3 = \Delta_{31} + \Delta_{32} + \Delta_{33} + \Delta_{34} + \Delta_{35} + \Delta_{36} + \Delta_{37}$ where

$$\Delta_{31} = \sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_i t_{i+1},$$

$$\Delta_{32} = \sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_i t_{i+2},$$

$$\Delta_{33} = \sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+1}t_{i+2},$$

$$\Delta_{34} = \sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_i t_{i+1},$$

$$\Delta_{35} = \sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_i t_{i+2},$$

$$\Delta_{36} = \sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+1}t_{i+2}$$

and

$$\Delta_{37} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} t_{j+1} s_j.$$

Now we notice $\Delta_{31} = \Delta_{32} = \Delta_{33} = \Delta_{34} = \Delta_{35} = \Delta_{36} = 0$ by (3.4.4) and $\Delta_{37} = 0$ by (3.4.3).

Now we also have $\Delta_4 = \Delta_{41} - \Delta_{42}$ where

$$\Delta_{41} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2}^2 (s_j + s_{j+1})$$

and

$$\Delta_{42} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (s_j + s_{j+1}).$$

Both these terms vanish by (3.4.3) and (3.3.6) respectively.

Thus we have shown $d_k = 0$ for the type I sequence. It is a matter of algebraic manipulation to show that $d_{k+1} = d_{k+2} = 0$. This algebra is very similar to that exhibited just before the statement of (3.3.7).

We move now on the analysis of loop type II. The type II loop begins

$$\begin{aligned} w_0 &= (1,0,0,0) \\ w_1 &= (0,1,0,0) \\ w_2 &= (0,0,0,0). \end{aligned}$$

We deal with the type II loop $(w_i) = (u_i, t_i, c_i, d_i)$. By similar argument

$$c_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(t_i u_{i+1} + u_{i+2}(t_i + t_{i+1})).$$

Use the equations (3.1) and (3.2) to see that

$$c_\alpha = \sum s_{\alpha-i-1}(s_{i+1} - s_i)s_i + \sum s_{\alpha-i-1} s_{i+1}(s_{i+2} - s_i),$$

which vanishes by (3.3.6).

We need a formula for d_α which is, by induction,

$$d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}u_{i+1}c_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}\binom{u_{i+1}}{2}t_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}u_{i+2}(c_i + c_{i+1} + u_{i+1}t_i)$$

$$+\sum_{i=0}^{\alpha-1} s_{\alpha-i-1}\binom{u_{i+2}}{2}(t_i + t_{i+1}).$$

We break up the expression for $d_k = \eta_1 + \eta_2 + \eta_3 + \eta_4$ where

$$\eta_1 = \sum s_{\alpha-i-1}u_{i+1}c_i,$$

$$\eta_2 = \sum s_{\alpha-i-1}\binom{u_{i+1}}{2}t_i,$$

$$\eta_3 = \sum s_{\alpha-i-1}u_{i+2}(c_i + c_{i+1} + u_{i+1}t_i)$$

and

$$\eta_4 = \sum s_{\alpha-i-1}\binom{u_{i+2}}{2}(t_i + t_{i+1}),$$

and we will now attempt to show that each of these four expressions η_i actually vanishes. The easy way to do this is to use (3.2) in order to write each η_i in terms of s_i, t_i and c_i. Thus we have

$$\eta_1 = \sum s_{\alpha-i-1}s_i c_i$$

$$\eta_2 = \frac{1}{2}\sum s_{\alpha-i-1}(s_i^2 - s_i)t_i$$

$$\eta_3 = \sum s_{\alpha-i-1}s_{i+1}(c_i + c_{i+1} + s_i t_i)$$

and

$$\eta_4 = \frac{1}{2}\sum s_{\alpha-i-1}(s_{i+1}^2 - s_{i+1})(t_i + t_{i+1}).$$

All η_i vanish by exactly the same argument with Δ_i and so we are done.

3.5 More Results

As a different perspective of what it used to be, k denotes the fundamental period of the standard r-step ($r = 2,3$) general Fibonacci sequences taken modulo a distinguished prime p.

Theorem 3.5.1 *Let G be a non-trivial finite $p-$group of exponent p and nilpotency class 2. Then $k(G) = k$.*

3.6 2−Step General Recurrences

We consider the 2−generator relatively free group G in the variety of nilpotent groups of class 2 and exponent p. We assume G is free on a set of generators $\{g_1, g_2\}$. This group G has order p^3 and we put $(g_2, g_1) = g_3$. The subgroup $< g_3 >$ has order p and is the centre of G. Every element of G has a unique representation as

$$g_1^a g_2^b g_3^c,$$

where the exponents are in $\mathbf{Z}/p\mathbf{Z}$. Let (s_i) denote the 2−step general recurrence defined by $s_i = ls_{i-1} + ms_{i-2}$ for some $l, m \in \mathbf{N}$. We assume that p does not divide m, and then we get the definition of 2−step general standard Fibonacci sequence which is $(0, 1, l, l^2 + m, \ldots)$ in $\mathbf{Z}/p\mathbf{Z}$. If p were permitted to divide m, then the sequence would ultimately be periodic, but would never return to the consecutive pair $0, 1$.

Now we have the formula for multiplication of two elements of the group. This is, by letting $x = g_1$, $y = g_2$ and $z = g_3$,

$$(x^{a_0} y^{b_0} z^{c_0}) \cdot (x^{a_1} y^{b_1} z^{c_1}) = (x^{a_2} y^{b_2} z^{c_2}),$$

where

$$a_2 = a_0 + a_1,$$

$$b_2 = b_0 + b_1$$

and

$$c_2 = c_0 + c_1 + a_1 b_0.$$

Let $(x^a y^b z^c)$ be an element of G. We claim that

$$(x^a y^b z^c)^n = (x^{na} y^{nb} z^{nc+\binom{n}{2}ab}). \tag{3.29}$$

We show this by induction on n. Since $\binom{1}{2} = 0$ our argument is obvious for $n = 1$. So assume that

$$(x^a y^b z^c)^{n-1} = (x^{(n-1)a} y^{(n-1)b} z^{(n-1)c+\binom{n-1}{2}ab}).$$

Multiplication of the both side by $(x^a y^b z^c)$ gives the left hand side of (3.29), and we have

$$(x^a y^b z^c)^n = (x^{(n-1)a} y^{(n-1)b} z^{(n-1)c+\binom{n-1}{2}ab}) \cdot (x^a y^b z^c).$$

Notice that z is in the centre of G, so that

$$(x^a y^b z^c)^n = (x^{(n-1)a} y^{(n-1)b} x^a y^b z^{nc+\binom{n-1}{2}ab}).$$

Since $z = (y, x)$, we have

$$(x^a y^b z^c)^n = (x^{na} y^{nb} z^{nc+\frac{1}{2}(n-1)(n-2)ab+(n-1)ab})$$

and so

$$(x^a y^b z^c)^n = (x^{na} y^{nb} z^{nc+(n-1)ab(1+\frac{n-2}{2})}).$$

Thus

$$(x^a y^b z^c)^n = (x^{na} y^{nb} z^{nc+\frac{1}{2}n(n-1)ab}),$$

and so we are done.

Let $(x^{a_0} y^{b_0} z^{c_0})$ and $(x^{a_1} y^{b_1} z^{c_1})$ be two elements in G. We get the formula for multiplication of powers of these two elements. This is, using (3.29),

$$(x^{a_0} y^{b_0} z^{c_0})^m \cdot (x^{a_1} y^{b_1} z^{c_1})^l = x^{a_2} y^{b_2} z^{c_2}$$

where

$$a_2 = ma_0 + la_1,$$

$$b_2 = mb_0 + lb_1$$

and

$$c_2 = mc_0 + lc_1 + lma_1b_0 + \binom{m}{2}a_0b_0 + \binom{l}{2}a_1b_1.$$

Define the bi-infinite sequence $(r_i) = (a_i, b_i, c_i)$ using the 2−step recurrence and initial data

$$r_0 = (1,0,0)$$

and

$$r_1 = (0,1,0).$$

By this time our auxiliary Fibonacci sequences are (s_i) and (t_i) by using the following initial data:

$$(s_0, s_1) = (0,1)$$

and

$$(t_0, t_1) = (1,0),$$

so that $(a_i) = (t_i)$ and $(b_i) = (s_i)$. Moreover equation (3.1) becomes

$$t_i = s_{i+1} - ls_i. \tag{3.30}$$

A straightforward induction yields that for $\alpha \geq 0$

$$c_\alpha = lm \sum s_{\alpha-i-1}t_{i+1}s_i + \binom{m}{2} \sum s_{\alpha-i-1}t_is_i + \binom{l}{2} \sum s_{\alpha-i-1}t_{i+1}s_{i+1}.$$

We use the equation (3.30) to get

$$c_\alpha = lm \sum s_{\alpha-i-1}s_{i+2}s_i - l^2m \sum s_{\alpha-i-1}s_{i+1}s_i + \binom{m}{2} \sum s_{\alpha-i-1}s_{i+1}s_i - \binom{m}{2}l \sum s_{\alpha-i-1}s_i^2$$

$$+\binom{l}{2} \sum s_{\alpha-i-1}s_{i+2}s_{i+1} - \binom{l}{2}l \sum s_{\alpha-i-1}s_{i+1}^2.$$

Lemma 3.6.1 *For all a and b we have*

$$\sum s_i s_{i+a} s_{-i+b} = 0. \tag{3.31}$$

Proof Let

$$c_{a,b} = \sum s_i s_{i+a} s_{-i+b}$$

where s_i is defined by the initial data $(0, 1)$ and the recurrence $s_i = ls_{i-1} + ms_{i-2}$. Now we claim that the quantities $c_{a,b}$ satisfy the *template equations.* These are as follows.

$$c_{a,b} = lc_{a-1,b} + mc_{a-2,b},$$

$$c_{a,b} = lc_{a,b-1} + mc_{a,b-2}$$

and

$$c_{a,b} = lc_{a+1,b-1} + mc_{a+2,b-2}.$$

These equations hold for all integers a and b. We need one more family of equations, since these alone are not enough to force the vanishing of the quantities $c_{a,b}$. Every loop is a rotation of the standard loop and its one place rotation $s^+ = s^1 = (s_1, s_2, \ldots)$. The loop s^α, the standard loop rotated through α places, can be written as

$$s_\alpha = s_\alpha s^+ + (s_{\alpha+1} - ls_\alpha)s.$$

Looking at position $\alpha + \beta$, this tells us that

$$s_{\alpha+\beta} = s_\alpha s_{\beta+1} + (s_{\alpha+1} - ls_\alpha)s_\beta. \tag{3.32}$$

Put $\alpha = i + a$ and $\beta = -i + b$ in (3.32), multiply both sides by s_i and sum over a fundamental range. The left hand side vanishes unless p is a divisor of $l + m - 1$. Thus we obtain another *template equation,* valid for all a and b, which is

$$c_{a,b+1} + c_{a+1,b} - lc_{a,b} = 0.$$

In order to see that the template equations are (modulo some more bad primes) enough to force each $c_{a,b}$ to vanish, we refer to the AXIOM code (cabstep2 code) and give the result in table 3.1 for $0 \le l \le 10$, $1 \le m \le 10$.

Corollary 3.6.2 *For all integers a, b and c we have*

$$\sum s_{i+a} s_{-i+b} s_{i+c} = 0. \tag{3.33}$$

All we need to do is to show that $c_\alpha = c_{\alpha+1} = 0$. We shall start with $c_\alpha = 0$. The fact that $c_{\alpha+1}$ also vanishes is an easy consequence of this. Now this is simply the upshot of (3.6.1) and (3.6.2).

The table 3.1 shows for the various values of l, m what we call "bad primes" for which we do not know whether $c_\alpha = 0$ or not. However, we do know, when $c_\alpha = 0$ for some l, m, that is the case for all primes except bad ones. The value of l, m are given in the first column of the table and corresponding bad primes in the second column. We demonstrate some good primes in the next row for the same l, m where $c_\alpha = 0$.

Different Wall numbers for the nilpotency class 1 and the nilpotency class 2 means that c_α is not zero for the certain values of l, m and the prime number p. Wall numbers separated by commas are those corresponding values for the prime numbers separated by commas respectively for some l, m. Since we work on modulo prime number p, Wall numbers are not given where one of the l, m is bigger than the prime number p.

3.7 3−Step General Recurrences

We agree on the group G to be exactly the same group in section 3.3. Let (s_i) denote the 3−step general recurrence defined by $s_i = l s_{i-1} + m s_{i-2} + n s_{i-3}$ for some $l, m, n \in \mathbb{N}$. Moreover let $(x^{a_0} y^{b_0} z^{c_0} t^{d_0} u^{e_0} w^{f_0})$, $(x^{a_1} y^{b_1} z^{c_1} t^{d_1} u^{e_1} w^{f_1})$ and $(x^{a_2} y^{b_2} z^{c_2} t^{d_2} u^{e_2} w^{f_2})$ be three elements of G, where $x = g_1$, $y = g_2$, $z = g_3$, $t = g_4$, $u = g_5$ and $w = g_6$. In a similar way to the proof by induction on n in previous section, we have that

$$(x^a y^b z^c t^d u^e w^f)^n = (x^{na} y^{nb} z^{nc} t^{nd+\binom{n}{2}ab} u^{ne+\binom{n}{2}ac} w^{nf+\binom{n}{2}bc}).$$

l,m	Primes	Wall number Nilp.cl. 1	Wall number Nilp.cl. 2
0,1			
	5,7,11	2,2,2	2,2,2
0,2	2		
	5,7,11	8,6,20	8,6,20
0,3	2,3		
	5,7,11	8,12,10	8,12,10
0,4	2,3		
	5,7,11	4,6,10	4,6,10
0,5	2,5		
	7,11,13	12,10,8	12,10,8
0,6	2,3,5		
	7,11,13	4,20,24	4,20,24
0,7	2,3,7		
	11,13,17	20,24,32	20,24,32
0,8	2,7		
	11,13,17	20,8,16	20,8,16
0,9	2,3		
	11,13,17	10,6,16	10,6,16
0,10	2,3,5		
	11,13,17	4,12,32	4,12,32
1,1	2	3	3
	7,13,17	16,28,36	16,28,36
1,2	2,5	4	4
	11,13,17	10,12,8	10,12,8
1,3	2,3		
	11,13,17	120,156,16	120,156,16
1,4	2		
	11,13,17	120,12,136	120,12,136
1,5	2,5		
	11,13,17	40,56,16	40,56,16
1,6	2,3,11	5	5
	19,23,29	18,22,28	18,22,28
1,7	2,5,7		
	19,23,29	120,11,812	120,11,812
1,8	2,31	30	30
	19,23,29	60,176,28	60,176,28
1,9	2,3,11	10	10
	19,23,29	72,528,210	72,528,210
1,10	2,5,59	29	29
	19,23,29	180,11,280	180,11,280

l,m	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
2,1	2,7	6	6
	19,23,29	40,22,20	40,22,20
2,2	2,3	3	3
	19,23,29	180,22,840	180,22,840
2,3	2,3,5	4	4
	19,23,29	18,22,28	18,22,28
2,4	2,5	5	5
	19,23,29	18,528,28	18,528,28
2,5	2,3,5		
	19,23,29	18,11,14	18,11,14
2,6	2,3,7	6	6
	31,37,41	30,12,560	30,12,560
2,7	2,7		
	31,37,41	30,684,40	30,684,40
2,8			
	31,37,41	10,36,20	10,36,20
2,9	2,3,5		
	31,37,41	30,18,40	30,18,40
2,10	2,5,11	11	11
	31,37,41	960,36,140	960,36,140
3,1	2,3		
	31,37,41	64,76,28	64,76,28
3,2	2,43	7	7
	31,37,41	320,1368,280	320,1368,280
3,3	2,3,5	4	20
	31,37,41	480,36,40	480,36,40
3,4	2,3,17	4	4
	31,37,41	10,18,10	10,18,10
3,5	2,5,7,13	3,12	21,12
	31,37,41	192,1368,840	192,1368,840
3,6	2,3,17	16	16
	43,47,53	264,2208,702	264,2208,702
3,7	2,3,7		
	43,47,53	33,46,52	33,46,52
3,8	2,5,43	21	21
	17,47,53	48,2208,2808	48,2208,2808
3,9	2,3,11	10	110
	43,47,53	1848,736,702	1848,736,702
3,10	2,3,5		
	43,47,53	42,46,52	42,46,52

l,m	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
4,1	2,19	6	6
	43,47,53	88,32,36	88,32,36
4,2	2,5,43	4,21	20,21
	31,47,53	320,46,52	320,46,52
4,3	2,3,47	46	46
	31,37,53	30,18,52	30,18,52
4,4	2,5,7	3,6	3,42
	43,47,53	616,46,351	616,46,351
4,5	2,5,13	4	4
	43,47,53	42,46,52	42,46,52
4,6	2,3,53	13	13
	41,43,47	40,42,2208	40,42,2208
4,7	2,5,7,53	13	13
	41,43,47	1680,21,2208	1680,21,2208
4,8	2,11,13	5,12	55,12
	43,47,53	308,46,936	308,46,936
4,9	2,3,5		
	43,47,53	42,2208,26	42,2208,26
4,10	2,5,13,47	3,23	39,23
	43,47,53	42,23,1404	42,23,1404
5,1	2,5,7	6	6
	43,47,53	88,96,52	88,96,52
5,2	2,3,17	8	8
	43,47,53	308,2208,2808	308,2208,2808
5,3	2,3,7,41	3,8	21,8
	59,61,67	3480,155,33	3480,155,33
5,4	2,173	43	43
	59,61,67	58,30,4488	58,30,4488
5,5	2,3,5		
	59,61,67	58,60,374	58,60,374
5,6	2,3,5,37	4	4
	59,61,67	58,60,66	58,60,66
5,7	2,7,11,47	5,46	55,46
	59,61,67	58,3720,2244	58,3720,2244
5,8	2,3,7		
	59,61,67	58,60,748	58,60,748
5,9	2,3,13,47	6,46	78,46
	59,61,67	3480,1220,1496	3480,1220,1496
5,10	2,5,7,37	18	18
	59,61,67	290,60,66	290,60,66

l,m	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
6,1	2,3,13	6	6
	59,61,67	40,124,22	40,124,22
6,2	2,5,7	6	42
	59,61,67	870,3720,561	870,3720,561
6,3	2,3,11	10	10
	59,61,67	58,30,748	58,30,748
6,4	2,3,23	22	22
	59,61,67	696,60,4488	696,60,4488
6,5	2,5,11,13	10,12	10,12
	71,73,79	720,5328,1248	720,5328,1248
6,6	2,3,7,11	3,5	3,55
	71,73,79	70,2664,3120	70,2664,3120
6,7	2,3,5,7		
	71,73,79	70,24,78	70,24,78
6,8	2,13,19	4,9	52,9
	71,73,79	5040,444,2080	5040,444,2080
6,9	2,3,7,17	8	8
	71,73,79	70,72,78	70,72,78
6,10	2,3,5,17	16	16
	71,73,79	70,72,78	70,72,78
7,1	2,7,13	6	6
	71,73,79	48,148,160	48,148,160
7,2	2,383	382	382
	71,73,79	70,72,1248	70,72,1248
7,3	2,3,5		
	71,73,79	5040,36,3120	5040,36,3120
7,4	2,5,83	82	82
	71,73,79	1680,72,78	1680,72,78
7,5	2,5,11,107	10,53	110,53
	83,89,97	82,44,9408	82,44,9408
7,6	2,3,439	73	73
	83,89,97	3444,88,96	3444,88,96
7,7	2,7,13	12	156
	83,89,97	82,880,9408	82,880,9408
7,8	2,5,7,13	4	4
	83,89,97	82,22,16	82,22,16
7,9	2,3,5,23	22	22
	83,89,97	6888,88,96	6888,88,96
7,10	2,5,463	231	231
	83,89,97	2296,7832,96	2296,7832,96

l,m	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
8,1	2,67	6	6
	83,89,97	82,88,196	82,88,196
8,2	2,3,31	30	30
	83,89,97	861,44,96	861,44,96
8,3	2,3,5,17	16	16
	83,89,97	2296,1584,4704	2296,1584,4704
8,4	2,11,149	10,37	110,37
	83,89,97	2296,44,2352	2296,44,2352
8,5	2,3,5,17	16	16
	83,89,97	82,88,9408	82,88,9408
8,6	2,3,13,313	12,52	156,52
	19,23,29	72,176,28	72,176,28
8,7	2,7,11,29	5,14	5,14
	19,23,31	18,253,960	18,253,960
8,8	2,3,5		
	19,23,29	18,22,28	18,22,28
8,9	2,3,41	4	4
	19,23,29	18,22,14	18,22,14
8,10	2,5,17,331	16,165	272,165
	19,23,29	18,22,840	18,22,840
9,1	2,3,7		
	19,23,29	18,22,60	18,22,60
9,2	2,5,11,71	5,70	5,70
	19,23,29	180,528,168	180,528,168
9,3	2,3,11,67	10,11	110,11
	19,23,29	9,22,28	9,22,28
9,4	2,3,5,11	10	10
	19,23,29	360,176,105	360,176,105
9,5	2,5,13,211	4,210	52,210
	19,23,29	18,11,70	18,11,70
9,6	2,3,7,41	40	40
	23,29,31	22,21,24	22,21,24
9,7	2,3,5,7,73	72	72
	23,29,31	264,28,30	264,28,30
9,8	2,7,127	42	42
	23,29,31	528,280,30	528,280,30
9,9	2,3,5,17	8	136
	23,29,31	22,14,960	22,14,960
9,10	2,3,5,101	4	4
	23,29,31	22,28,30	22,28,30

l,m	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
10,1	2,5,103	6	6
	23,29,31	22,60,64	22,60,64
10,2	2,11,23	5,22	55,22
	19,29,31	180,280,320	180,280,320
10,3	2,3,271	45	45
	23,29,31	528,28,15	528,28,15
10,4	2,13,277	3,69	39,69
	23,29,31	22,406,320	22,406,320
10,5	2,5,7,113	112	112
	23,29,31	132,14,192	132,14,192
10,6	2,3,5,23	22	22
	19,29,31	360,210,93	360,210,93
10,7	2,7,73	72	72
	23,29,31	11,420,30	11,420,30
10,8	2,17,37	8,36	136,36
	23,29,31	528,28,30	528,28,30
10,9	2,3,599	598	598
	23,29,31	528,7,960	528,7,960
10,10	2,5,11,19	3,9	3,171
	23,29,31	22,28,30	22,28,30

Table 3.1:

We have the formula for product of three elements of the group

$$(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0}w^{f_0})^n \cdot (x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1}w^{f_1})^m \cdot (x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2}w^{f_2})^l = (x^{a_3}y^{b_3}z^{c_3}t^{d_3}u^{e_3}w^{f_3})$$

where

$$a_3 = na_0 + ma_1 + la_2,$$

$$b_3 = nb_0 + mb_1 + lb_2,$$

$$c_3 = nc_0 + mc_1 + lc_2,$$

$$d_3 = nd_0 + md_1 + ld_2 + \binom{n}{2}a_0b_0 + \binom{m}{2}a_1b_1 + \binom{l}{2}a_2b_2 + mna_1b_0 + la_2(nb_0 + mb_1),$$

$$e_3 = ne_0 + me_1 + le_2 + \binom{n}{2}a_0c_0 + \binom{m}{2}a_1c_1 + \binom{l}{2}a_2c_2 + mna_1c_0 + la_2(nc_0 + mc_1),$$

and

$$f_3 = nf_0 + mf_1 + lf_2 + \binom{n}{2}b_0c_0 + \binom{m}{2}b_1c_1 + \binom{l}{2}b_2c_2 + mnb_1c_0 + lb_2(nc_0 + mc_1).$$

We preserve the bi-infinite sequence (r_i) and auxiliary sequences $(s_i), (u_i)$ and (t_i) of section 3.3. One must notice that both (t_i) and (u_i) can be expressed in terms of (s_i). In particular, we have, for each i, the pair of equations

$$t_i = s_{i+1} - ls_i \tag{3.34}$$

and

$$u_i = s_{i+2} - ls_{i+1} - ms_i. \tag{3.35}$$

A straightforward induction yields that for $\alpha \geq 0$

$$d_\alpha = \binom{n}{2}\sum s_{\alpha-i-1}s_it_i + \binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}t_{i+1} + \binom{l}{2}\sum s_{\alpha-i-1}s_{i+2}t_{i+2}$$

$$+mn\sum s_{\alpha-i-1}t_{i+1}s_i + ln\sum s_{\alpha-i-1}t_{i+2}s_i + lm\sum s_{\alpha-i-1}t_{i+2}s_{i+1},$$

$$e_\alpha = \binom{n}{2}\sum s_{\alpha-i-1}t_iu_i + \binom{m}{2}\sum s_{\alpha-i-1}t_{i+1}u_{i+1} + \binom{l}{2}\sum s_{\alpha-i-1}t_{i+2}u_{i+2}$$

$$+mn\sum s_{\alpha-i-1}t_{i+1}u_i + ln\sum s_{\alpha-i-1}t_{i+2}u_i + lm\sum s_{\alpha-i-1}t_{i+2}u_{i+1}$$

and

$$f_\alpha = \binom{n}{2}\sum s_{\alpha-i-1}s_iu_i + \binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}u_{i+1} + \binom{l}{2}\sum s_{\alpha-i-1}s_{i+2}u_{i+2}$$

$$+mn\sum s_{\alpha-i-1}s_{i+1}u_i + ln\sum s_{\alpha-i-1}s_{i+2}u_i + lm\sum s_{\alpha-i-1}s_{i+2}u_{i+1}.$$

We use the equations (3.34) and (3.35) to get d_α, e_α and f_α depending on just s_i.

Lemma 3.7.1 *For all a and b we have*

$$\sum s_is_{i+a}s_{-i+b} = 0. \tag{3.36}$$

Proof Let

$$c_{a,b} = \sum s_is_{i+a}s_{-i+b}$$

where $s_i = ls_{i-1} + ms_{i-2} + ns_{i-3}$. The *template equations* are much as before, and for identical reasons. They are

$$c_{a,b} = lc_{a-1,b} + mc_{a-2,b} + nc_{a-3,b},$$

$$c_{a,b} = lc_{a,b-1} + mc_{a,b-2} + nc_{a,b-3},$$

$$c_{a,b} = lc_{a+1,b-1} + mc_{a+2,b-2} + nc_{a+3,b-3}$$

and

$$c_{a,b+2} + c_{a+1,b+1} - lc_{a,b+1} + c_{a+2,b} - lc_{a+1,b} - mc_{a,b} = 0.$$

The first three equations are easy consequences of the recurrence relation. The fourth is obtained in a similar way to the last template equation in the previous section. This collection of 4 systems of equations is enough to force each $c_{a,b}$ to vanish save at finitely many bad primes (which are a function of the particular recurrence coefficients). To see this, we apply AXIOM code, (see cabstep3 code), and we show the results in table 3.2 for $0 \le l, m \le 5$ and $1 \le n \le 5$.

Corollary 3.7.2 *For all integers a, b and c we have*

$$\sum s_{i+a} s_{-i+b} s_{i+c} = 0. \tag{3.37}$$

Our aim is to show

$$d_\alpha = d_{\alpha+1} = d_{\alpha+2} = e_\alpha = e_{\alpha+1} = e_{\alpha+2} = f_\alpha = f_{\alpha+1} = f_{\alpha+2} = 0.$$

We shall start with $d_\alpha = e_\alpha = f_\alpha = 0$. In fact this is simply the upshot of (3.7.1) and (3.7.2). The other terms $d_{\alpha+1}, d_{\alpha+2}$ etc, are also vanish by the same argument.

The table 3.2 is to be read in the same way of the table 3.1 and gives when $d_\alpha = e_\alpha = f_\alpha = 0$ and, of course those, exclusive primes for which we do not know whether d_α, e_α and f_α are zero or not. Thus we have proved (3.5.1).

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
0,0,1			
	7,11,13	3,3,3	3,3,3
0,0,2	2		
	7,11,13	9,30,36	9,30,36
0,0,3	2,3		
	7,11,13	18,15,9	18,15,9
0,0,4	2,3		
	7,11,13	9,15,18	9,15,18
0,0,5	2,5		
	7,11,13	18,15,12	18,15,12
0,1,1	2	7	7
	7,11,13	48,120,183	48,120,183
0,1,2	2,13	156	156
	5,7,11	124,171,120	124,171,120
0,1,3	2,3		
	7,11,13	24,665,168	24,665,168
0,1,4	2,5,47	24,46	120,46
	7,11,13	24,665,1098	24,665,1098
0,1,5	2,5,11	55	55
	7,13,17	342,168,4912	342,168,4912
0,2,1			
	5	20	20
0,2,2	2,3,23	8,22	24,22
	5,7,11	124,171,1330	124,171,1330
0,2,3	2,3,53	52	52
	5,7,11	124,24,665	124,24,665
0,2,4			
	5	20	20
0,2,5	2,3,5,191	190	190
	7,11,13	342,60,168	342,60,168
0,3,1	2,3		
	7,11,13	57,133,183	57,133,183
0,3,2	2,5	20	20
	7,11,13	42,110,183	42,110,183
0,3,3	2,3,5	20	20
	7,11,13	48,120,168	48,120,168
0,3,4	2,3,41	40	40
	7,11,13	48,120,1098	48,120,1098
0,3,5	2,5,7,23	42,22	42,22
	11,13,17	665,168,96	665,168,96

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
0,4,1	2,37	36	36
	7,11,13	48,133,56	48,133,56
0,4,2	2,5,67	24,66	120,66
	7,11,13	171,1330,168	171,1330,168
0,4,3	2,3,17	16	16
	7,11,13	24,120,156	24,120,156
0,4,4	2,7,53	24,26	168,26
	5,11,13	62,55,84	62,55,84
0,4,5	2,5,107	106	106
	7,11,13	342,120,244	342,120,244
0,5,1	2,3,5,11	55	55
	7,13,17	19,84,288	19,84,288
0,5,2	2,3,7,17	42,16	42,16
	11,13,19	120,84,360	120,84,360
0,5,3	2,3,7,113	48,112	336,112
	11,13,17	665,549,4912	665,549,4912
0,5,4	2,3,13	12	12
	7,11,17	48,120,136	48,120,136
0,5,5	2,3,5,7	42	42
	11,13,17	665,168,144	665,168,144
1,0,1	2,3	7,8	7,24
	5,7,11	31,57,60	31,57,60
1,0,2	2,19	18	18
	5,7,11	124,48,1330	124,48,1330
1,0,3	2,3,5	20	20
	7,11,13	342,665,84	342,665,84
1,0,4			
	7	42	42
1,0,5	2,5,17	16	16
	7,11,13	24,665,168	24,665,168
1,1,1	2	4	4
	5,7,11	31,48,110	31,48,110
1,1,2			
	3,5,7	6,12,21	6,12,21
1,1,3	2,3		
	5,7,11	124,342,665	124,342,665
1,1,4	2,5,7	10,48	10,48
	11,13,17	60,168,288	60,168,288
1,1,5	2,3,5		
	7,11,13	342,35,156	342,35,156

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
1,2,1	2,3	4	12
	5,7,11	31,57,30	31,57,30
1,2,2	2,37	36	36
	5,7,11	12,171,120	12,171,120
1,2,3	2,3,5,11	8,10	40,110
	7,13,17	342,168,4912	342,168,4912
1,2,4	2,3,37	36	36
	5,7,11	62,171,665	62,171,665
1,2,5	2,5,7,11	6,40	42,440
	13,17,19	84,144,3429	84,144,3429
1,3,1			
1,3,2	2,5,7,83	8,48,82	40,336,82
	11,13,17	190,2196,48	190,2196,48
1,3,3	2,3,37	36	36
	11,13,17	15,549,144	15,549,144
1,3,4	2,3,5,7,149	24,12,74	120,84,74
	11,13,17	665,1098,96	665,1098,96
1,3,5	2,5,53	2756	2756
	11,13,17	60,244,4912	60,244,4912
1,4,1	2,5,47	4,23	20,23
	11,13,17	133,39,307	133,39,307
1,4,2	2,3,13	12	12
	5,7,11	24,48,10	24,48,10
1,4,3	2,3,7,11,131	6,15,130	42,165,130
	5,13,17	124,549,144	124,549,144
1,4,4	2,19	9	9
	5,7,11	62,24,665	62,24,665
1,4,5	2,3,5,19	360	6840
	11,13,17	665,732,4912	665,732,4912
1,5,1	2,3,37	36	36
	11,13,17	120,183,307	120,183,307
1,5,2	2,7,53	12,52	84,52
	11,13,17	1330,168,2456	1330,168,2456
1,5,3	2,3,5		
	11,13,17	1330,78,272	1330,78,272
1,5,4	2,3,5,607	606	606
	11,13,17	665,1098,288	665,1098,288
1,5,5	2,5,13	12	12
	7,11,17	24,120,4912	24,120,4912

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
2,0,1	2,17	16	16
	7,11,13	19,120,168	19,120,168
2,0,2	2,3,7,59	8,48,58	24,336,58
	5,11,13	24,1330,2196	24,1330,2196
2,0,3	2,3,47	23	23
	5,11,13	124,120,549	124,120,549
2,0,4	2,5,71	20,70	20,70
	7,11,13	42,665,1098	42,665,1098
2,0,5	2,3,5,103	102	102
	7,11,13	24,665,732	24,665,732
2,1,1	2,3	6	18
	7,11,13	57,19,61	57,19,61
2,1,2	2,17	16	272
	7,11,13	171,30,168	171,30,168
2,1,3	2,3,5,23	4,264	20,6072
	7,11,13	342,665,549	342,665,549
2,1,4	2,3,83	82	82
	7,11,13	171,60,168	171,60,168
2,1,5	2,5,7,19,29	6,171,840	42,171,840
	11,13,17	95,28,4912	95,28,4912
2,2,1	2,5	8	8
	7,11,13	57,133,168	57,133,168
2,2,2	2,3,5	4,24	4,120
	7,11,13	48,120,84	48,120,84
2,2,3			
	7,11,13	6,15,39	6,15,39
2,2,4	2,7	24	168
	5,11,13	10,665,168	10,665,168
2,2,5	2,3,5		
	7,11,13	342,120,732	342,120,732
2,3,1	2,5	6	30
	7,11,13	12,30,183	12,30,183
2,3,2	2,3,17,211	288,210	4896,210
	7,11,13	171,1330,2196	171,1330,2196
2,3,3	2,3,7,13	48,12	336,12
	5,11,17	124,95,16	124,95,16
2,3,4	2,5,7,31	8,48,30	40,336,30
	11,13,17	120,1098,1228	120,1098,1228
2,3,5	2,3,5,41	40	1640
	7,11,13	342,10,244	342,10,244

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
2,4,1			
	13,	52,	52,
2,4,2	2,7,359	21,179	21,179
	11,13,17	1330,2196,2456	1330,2196,2456
2,4,3	2,3,13,353	12,352	156,352
	7,11,17	342,110,4912	342,110,4912
2,4,4	2,3,29	28	28
	7,11,13	48,665,1098	48,665,1098
2,4,5	2,5,11,73	30,72	330,72
	7,13,17	342,156,288	342,156,288
2,5,1	2,7,11,31	6,10,15	42,10,15
	13,17,19	183,307,57	183,307,57
2,5,2	2,3,43	14	14
	7,11,13	48,30,12	48,30,12
2,5,3	2,3,23,47	176,46	4048,46
	7,11,13	342,120,24	342,120,24
2,5,4	2,5,7,1823	12,911	84,911
	11,13,17	40,1098,272	40,1098,272
2,5,5	2,3,5,11	5	55
	7,13,17	48,732,4912	48,732,4912
3,0,1	2,3,5	20	20
	7,11,13	48,40,168	48,40,168
3,0,2	2,7,509	24,508	168,508
	5,11,13	124,120,2196	124,120,2196
3,0,3	2,3,5,13,29	24,84,28	120,1092,28
	7,11,17	42,665,144	42,665,144
3,0,4	2,3,103	102	102
	7,11,13	171,665,168	171,665,168
3,0,5	2,5,7,11,151	48,60,150	336,660,150
	13,17,19	168,4912,	
3,1,1	2,11	55	55
	5,7,13	31,24,84	31,24,84
3,1,2	2,5,31	4,160	20,4960
	7,11,13	171,30,2196	171,30,2196
3,1,3	2,3,11,31	120,30	1320,30
	5,7,13	124,24,549	124,24,549
3,1,4	2,5,7,11	24,6,110	120,42,110
	13,17,19	1098,1228,3429	1098,1228,3429
3,1,5	2,5,59	29	29
	7,11,13	48,120,732	48,120,732

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
3,2,1	2,3,5	5	25
	7,11,13	42,133,56	42,133,56
3,2,2	2,3,67	66	66
	7,11,13	171,1330,2196	171,1330,2196
3,2,3	2,3,7,31,47	48,30,368	336,30,368
	5,11,13	24,120,549	24,120,549
3,2,4	2,3,5,167	8,166	40,166
	7,11,13	171,665,42	171,665,42
3,2,5	2,3,5,71,193	360,192	360,192
	7,11,13	42,20,12	42,20,12
3,3,1	2,3,5	8	8
	7,11,13	19,40,61	19,40,61
3,3,2	2,7,11,31	24,10,320	168,10,320
	5,13,17	124,2196,272	124,2196,272
3,3,3	2,3		
	5,7,11	124,342,665	124,342,665
3,3,4			
	5,7,11	6,21,15	6,21,15
3,3,5	2,5,7,19	12,18	12,18
	11,13,17	665,244,272	665,244,272
3,4,1	2,7	7	7
	11,13,17	133,12,307	133,12,307
3,4,2	2,7,151	6,75	42,75
	11,13,17	30,84,2456	30,84,2456
3,4,3	2,3,47,61	2208,30	2208,30
	7,11,13	342,120,549	342,120,549
3,4,4	2,5,227	12,226	60,226
	7,11,13	171,665,168	171,665,168
3,4,5	2,5,11,73,79	60,72,78	660,72,78
	7,13,17	342,732,16	342,732,16
3,5,1			
3,5,2	2,3,23,1367	528,683	528,683
	7,11,13	171,190,2196	171,190,2196
3,5,3	2,3,5,11,1187	60,593	660,593
	7,13,17	48,168,288	48,168,288
3,5,4	2,3,7,11,19,53	48,15,18,1378	336,165,18,
	13,17,23	1098,288,253	1098,288,253
3,5,5	2,3,5,43	42	42
	7,11,13	342,120,84	342,120,84

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
4,0,1	2,3,83	41	41
	7,11,13	57,133,61	57,133,61
4,0,2	2,5,13,23	20,12,22	20,12,506
	7,11,17	48,120,2456	48,120,2456
4,0,3	2,3,11,23	10,11	110,11
	5,7,13	124,342,549	124,342,549
4,0,4	2,3,7,653	42,326	42,326
	5,11,13	62,40,156	62,40,156
4,0,5	2,5,199	99	99
	7,11,13	24,665,84	24,665,84
4,1,1	2,5,19	10,18	10,18
	7,11,13	57,120,12	57,120,12
4,1,2	2,3,7,37	48,36	336,36
	5,11,13	124,190,2196	124,190,2196
4,1,3	2,3,7,71	6,2485	42,2485
	5,11,13	12,95,168	12,95,168
4,1,4	2,7,13,863	24,168,862	168,2184,862
	5,11,17	62,665,96	62,665,96
4,1,5	2,3,5,17,109	272,540	272,
	7,11,13	342,665,168	342,665,168
4,2,1	2,3,139	23	
	7,11,13	48,120,168	48,120,168
4,2,2	2,5,7,29	8,21,812	40,21,812
	11,13,17	120,84,16	120,84,16
4,2,3	2,3,37,409	1368,	
	7,11,13	342,20,168	342,20,168
4,2,4	2,3,101	50,	
	7,11,13	48,55,1098	48,55,1098
4,2,5	2,5,29,39,47	840,	
	7,11,13	342,120,24	342,120,24
4,3,1	2,3,7,193	8,	56,
	5,11,13	24,120,78	24,120,78
4,3,2	2,7,97	12,48	84,48
	5,11,13	20,1330,84	20,1330,84
4,3,3	2,3,5,17	8,16	40,16
	7,11,13	42,665,549	42,665,549
4,3,4	2,3,5,11,29	12,55,28	60,55,28
	7,13,17	171,1098,48	171,1098,48
4,3,5	2,5,11,211	110,105	110,
	7,13,17	24,244,32	24,244,32

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
4,4,1	2,17	8	8
	5,7,11	31,19,60	31,19,60
4,4,2	2,3,23	11	11
	5,7,11	124,171,1330	124,171,1330
4,4,3	2,3,5,19,37	18,1368	18,1368
	7,11,13	342,665,168	342,665,168
4,4,4	2,5,11	4,10	4,110
	7,13,17	171,1098,96	171,1098,96
4,4,5			
	7,11,13	6,15,12	6,15,12
4,5,1	2,3,11	7	7
	7,13,17	12,61,307	12,61,307
4,5,2	2,5,41,821	240,	9840,
	7,11,13	171,110,168	171,110,168
4,5,3	2,3,11,17,43,71	60,16,42,70	660,16,42,4970
	7,13,19	48,168,6858	48,168,6858
4,5,4	2,3,11,937	40,	440,
	7,13,17	48,42,288	48,42,288
4,5,5	2,5,13,53	56,52	728,52
	7,11,17	342,665,4912	342,665,4912
5,1,1	2,3,199		
	7,11,13	21,15,183	21,15,183
5,1,2	2,7,53,71	6,52,70	42,52,4970
	11,13,17	40,84,2456	40,84,2456
5,1,3	2,3,13,173	42,	546,
	7,11,17	342,40,288	342,40,288
5,1,4	2,3,5,503		
	7,11,13	171,665,168	171,665,168
5,1,5	2,5,37,41	36,1680	36,
	7,11,13	342,665,168	342,665,168
5,2,1	2,5,7	8	56
	11,13,17	20,61,48	20,61,48
5,2,2	2,3,13	12	12
	7,11,17	171,60,288	171,60,288
5,2,3	2,3,107		
	7,11,13	24,665,24	24,665,24
5,2,4	2,5,7,367	6,	42,
	11,13,17	665,1098,96	665,1098,96
5,2,5	2,3,5,11,19,29	30,18,28	330,342,28
	7,13,17	48,244,4912	48,244,4912

l,m,n	Primes	Wall number Nilp.cl.1	Wall number Nilp.cl.2
5,3,1	2,173	172	
	7,11,13	48,133,183	48,133,183
5,3,2	2,3,5,71,97	1680,48	,48
	7,11,13	171,1330,168	171,1330,168
5,3,3	2,3,5,37	666	666
	7,11,13	48,40,549	48,40,549
5,3,4	2,11,13,977	110,168,	110,2184
	7,17,19	48,68,36	48,68,36
5,3,5	2,3,5,11,229	60,228	660,
	7,13,17	342,84,144	342,84,144
5,4,1	2,3,307		
	7,11,13	42,110,183	42,110,183
5,4,2	2,5,17,2131	16,	272,
	7,11,13	171,10,168	171,10,168
5,4,3	2,3,11,23,107	55,22,	55,22,
	7,13,17	48,549,4912	48,549,4912
5,4,4	2,3,79	78	
	7,11,13	171,120,168	171,120,168
5,4,5	2,5,13,179	12,	156,
	7,11,17	42,665,4912	42,665,4912
5,5,1	2,5,13	8	8
	7,11,17	57,133,288	57,133,288
5,5,2	2,3,7,11	48,120	48,1320
	13,17,19	2196,2456,72	2196,2456,72
5,5,3	2,3,7,13	12,84	12,84
	11,17,19	665,4912,180	665,4912,180
5,5,4	2,11,13,29	60,156,28	60,156,28
	7,17,19	171,1228,176	171,1228,176
5,5,5	2,3,5,7	48	336
	11,13,17	95,732,4912	95,732,4912

Table 3.2:

Chapter 4

General Recurrence Relations in Groups

We now move on to discuss two and three step linear recurrences in $p-$groups of nilpotency class 3. We are not able to prove general theorems of the form: *For an arbitrary linear recurrence Wall's number behaves well for these groups, except possibly at finitely many bad primes* (i.e. co-incides with the period of the standard sequence in the integers modulo p.)

Instead we present a method which, when applied to any specific recurrence, seems to yield the relevant theorems about $p-$groups. There are three basic obstructions to proving the theorems for all recurrences at once.

Obstruction 1: Certain elementary sums must vanish.

Obstruction 2: During the course of the method, we have to examine an infinitely large system of linear equations. It appears to suffice to take a finite subset of the equations, but, *a priori,* it is not clear that the "same" subset of the linear equations will suffice independent of the coefficients of the recurrence. It is certainly true that the longer the recurrence (the step-length), the more equations are needed.

Obstruction 3: For convenience we work with a quotient of the relevant relatively free group, as before. We prove theorems about the periods of recurrences in this quotient group, and then deduce that these theorems carry over to the relatively free group.

It is this final deduction which causes difficulty. The recurrence in question defines an automorphism of the relatively free group. Different recurrences (usually) define different automorphisms. The process of moving from the quotient to the relatively free group involves a detailed understanding of this automorphism, and how it acts on the kernels of maps to the quotient. This appears to be problematic in general, but easy enough in any specific case (i.e. when the recurrence is specified).

4.1 General 2–Step Recurrences

Recall that the defining equation is

$$s_i = ls_{i-1} + ms_{i-2}.$$

There is an associated sequence satisfying the same recurrence, defined by,

$$t_i = s_{i+1} - ls_i.$$

We work with the (familiar) group H where the composition law for elements is

$$(a,b,c,d)^m \cdot (a',b',c',d')^l = (a'',b'',c'',d'')$$

where

$$a'' = ma + la',$$

$$b'' = mb + lb',$$

$$c'' = mc + lc' + \binom{m}{2}ab + \binom{l}{2}a'b' + lma'b$$

and

$$d'' = md + ld' + \binom{m}{2}\binom{a}{2}b + \binom{m}{2}ac + \binom{m}{3}a^2b + \binom{l}{2}\binom{a'}{2}b' + \binom{l}{2}a'c'$$

$$+\binom{l}{3}a'^2b' + lma'c + l\binom{m}{2}aa'b + \binom{la'}{2}mb.$$

Thus, we have

$$d'' = \binom{m}{2}\sum s_{k-i-1}\binom{t_i}{2}s_i + \binom{m}{2}\sum s_{k-i-1}t_ic_i + \binom{m}{3}\sum s_{k-i-1}t_i^2s_i$$

$$+\binom{l}{2}\sum s_{k-i-1}\binom{t_{i+1}}{2}s_{i+1} + \binom{l}{2}\sum s_{k-i-1}t_{i+1}c_{i+1} + \binom{l}{3}\sum s_{k-i-1}t_{i+1}^2s_{i+1}$$

$$+lm\sum s_{k-i-1}t_{i+1}c_i + l\binom{m}{2}\sum s_{k-i-1}t_it_{i+1}s_i + m\sum s_{k-i-1}\binom{lt_{i+1}}{2}s_i$$

where

$$c_j = \binom{m}{2}\sum s_{j-i-1}t_is_i + \binom{l}{2}\sum s_{j-i-1}t_{i+1}s_{i+1} + lm\sum s_{j-i-1}t_{i+1}s_i.$$

We need to show d_k and d_{k+1} vanish. We shall only demonstrate the argument for d_k, the other case being entirely similar.

$$d_k = \binom{m}{2}\sum s_{k-i-1}\binom{t_i}{2}s_i + \binom{m}{2}\sum_j s_{k-j-1}t_j(\binom{m}{2}\sum s_{j-i-1}t_is_i + \binom{l}{2}\sum s_{j-i-1}t_{i+1}s_{i+1}$$

$$+lm\sum s_{j-i-1}t_{i+1}s_i) + \binom{m}{3}\sum s_{k-i-1}t_i^2s_i + \binom{l}{2}\sum s_{k-i-1}\binom{t_{i+1}}{2}s_{i+1} + \binom{l}{2}\sum s_{k-j-1}t_{j+1}$$

$$(\binom{m}{2}\sum s_{j-i-1}t_is_i + \binom{l}{2}\sum s_{j-i}t_{i+1}s_{i+1} + lm\sum s_{j-i}t_{i+1}s_i) + \binom{l}{3}\sum s_{k-i-1}t_{i+1}^2s_{i+1}$$

$$+lm\sum s_{k-j-1}t_{j+1}(\binom{m}{2}\sum s_{j-i-1}t_is_i + \binom{l}{2}\sum s_{j-i-1}t_{i+1}s_{i+1} + lm\sum s_{j-i-1}t_{i+1}s_i)$$

$$+l\binom{m}{2}\sum s_{k-i-1}t_it_{i+1}s_i + m\sum s_{k-i-1}\binom{lt_{i+1}}{2}s_i.$$

We can split up d_k into 15 sums d_i where $1 \leq i \leq 15$, and $d_k = \sum_{i=1}^{15} d_i$. The reader is urged to overlook the ambiguity in notation, which in this instance is harmless.

$$d_1 = \binom{m}{2}\sum s_{k-i-1}\binom{t_i}{2}s_i,$$

$$d_2 = \binom{m}{2}^2\sum_{i<j} s_{k-j-1}t_js_{j-i-1}t_is_i,$$

$$d_3 = \binom{m}{2}\binom{l}{2}\sum_{i<j} s_{k-j-1}t_js_{j-i-1}t_{i+1}s_{i+1},$$

$$d_4 = \binom{m}{2}lm\sum_{i<j} s_{k-j-1}t_js_{j-i-1}t_{i+1}s_i,$$

$$d_5 = \binom{m}{3} \sum s_{k-i-1} t_i^2 s_i,$$

$$d_6 = \binom{l}{2} \sum s_{k-i-1} \binom{t_{i+1}}{2} s_{i+1},$$

$$d_7 = \binom{l}{2}\binom{m}{2} \sum_{i<j} s_{k-j-1} t_{j+1} s_{j-i} t_i s_i,$$

$$d_8 = \binom{l}{2}^2 \sum_{i<j} s_{k-j-1} t_{j+1} s_{j-i} t_{i+1} s_{i+1},$$

$$d_9 = \binom{l}{2} lm \sum_{i<j} s_{k-j-1} t_{j+1} s_{j-i} t_{i+1} s_i,$$

$$d_{10} = \binom{l}{3} \sum s_{k-i-1} t_{i+1}^2 s_{i+1},$$

$$d_{11} = \binom{m}{2} lm \sum_{i<j} s_{k-j-1} t_{j+1} s_{j-i-1} t_i s_i,$$

$$d_{12} = \binom{l}{2} lm \sum_{i<j} s_{k-j-1} t_{j+1} s_{j-i-1} t_{i+1} s_{i+1},$$

$$d_{13} = l^2 m^2 \sum_{i<j} s_{k-j-1} t_{j+1} s_{j-i-1} t_{i+1} s_i,$$

$$d_{14} = \binom{m}{2} l \sum s_{k-i-1} t_i t_{i+1} s_i$$

and

$$d_{15} = m \sum s_{k-i-1} \binom{lt_{i+1}}{2} s_i.$$

We address each sum separately, with relentless enthusiasm. First we consider d_1. Ignore the "bad prime" 2. Now

$$\sum s_{k-i-1}(t_i{}^2 - t_i)s_i = \sum s_{k-i-1}((s_{i+1} - ls_i)^2 - (s_{i+1} - ls_i))s_i$$

which is a linear combination of

$$\sum s_{k-i-1} s_{i+1}^2 s_i, \ \sum s_{k-i-1} s_{i+1} s_i^2,$$

$$\sum s_{k-i-1} s_i^3, \ \sum s_{k-i-1} s_{i+1} s_i$$

and

$$\sum s_{k-i-1}s_i^2.$$

Now for d_2. A similar analysis shows that, ignoring the bad prime 2, d_2 is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}s_j s_{j-i-1}s_{i+1}s_i$$

and

$$\sum_{i<j} s_{k-j-1}s_j s_{j-i-1}s_i^2.$$

Now for d_3. A similar analysis shows that, ignoring the bad prime 2, d_3 is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}s_j s_{j-i-1}s_{i+2}s_{i+1}$$

and

$$\sum_{i<j} s_{k-j-1}s_j s_{j-i-1}s_{i+1}^2.$$

Now for d_4. A similar analysis shows that, ignoring the bad prime 2, d_4 is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_j s_{j-i-1}s_{i+2}s_i$$

and

$$\sum_{i<j} s_{k-j-1}s_js_{j-i-1}s_{i+1}s_i.$$

Now for d_5. A similar analysis shows that, ignoring the bad primes 2 and 3, d_5 is a linear combination of

$$\sum s_{k-i-1}s_{i+1}^2s_i, \sum s_{k-i-1}s_{i+1}s_i^2$$

and

$$\sum s_{k-i-1}s_i^3.$$

Now for d_6. A similar analysis shows that, ignoring the bad prime 2, d_6 is a linear combination of

$$\sum s_{k-i-1}s_{i+2}^2s_{i+1}, \sum s_{k-i-1}s_{i+2}s_{i+1}^2,$$

$$\sum s_{k-i-1}s_{i+1}^3,$$

$$\sum s_{k-i-1}s_{i+2}s_{i+1}$$

and

$$\sum s_{k-i-1}s_{i+1}^2.$$

Now for d_7. A similar analysis shows that, ignoring the bad prime 2, d_7 is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i}s_{i+1}s_i$$

and

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i}s_i^2.$$

Now for d_8. A similar analysis shows that, ignoring the bad prime 2, d_8 is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i}s_{i+2}s_{i+1}$$

and

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i}s_{i+1}^2.$$

Now for d_9. A similar analysis shows that, ignoring the bad prime 2, d_9 is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i}s_{i+2}s_i$$

and

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i}s_{i+1}s_i.$$

Now for d_{10}. A similar analysis shows that, ignoring the bad primes 2 and 3, d_{10} is a linear combination of

$$\sum s_{k-i-1}s_{i+2}^2s_{i+1},$$

$$\sum s_{k-i-1}s_{i+2}s_{i+1}^2$$

and

$$\sum s_{k-i-1}s_{i+1}^3.$$

Now for d_{11}. A similar analysis shows that, ignoring the bad prime 2, d_{11} is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i-1}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+1}s_i$$

and

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_i^2.$$

Now for d_{12}. There is no bad prime in this case, and d_{12} is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i-1}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i-1}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+2}s_{i+1}$$

and

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+1}^2.$$

Now for d_{13}. There is no bad prime in this case, and d_{13} is a linear combination of

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i-1}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+2}s_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+2}s_i$$

and

$$\sum_{i<j} s_{k-j-1}s_{j+1}s_{j-i-1}s_{i+1}s_i.$$

Now for d_{14}. A similar analysis shows that, ignoring the bad prime 2, d_{14} is a linear combination of

$$\sum s_{k-i-1}s_{i+1}s_{i+2}s_i,$$

$$\sum s_{k-i-1}s_{i+1}^2s_i,$$

$$\sum s_{k-i-1}s_i^2s_{i+2}$$

and

$$\sum s_{k-i-1}s_i^2 s_{i+1}.$$

Now for d_{15}. A similar analysis shows that, ignoring the bad prime 2, d_{15} is a linear combination of

$$\sum s_{k-i-1}s_{i+2}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+1}s_{i+2}s_i,$$

$$\sum s_{k-i-1}s_{i+1}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+2}s_i$$

and

$$\sum s_{k-i-1}s_{i+1}s_i.$$

We now address the question as to why each of these sums will vanish.

Lemma 4.1.1

$$\sum s_i = 0 \tag{4.1}$$

$$\sum s_i s_{i-1} = 0 \tag{4.2}$$

$$\sum s_i^2 = 0 \tag{4.3}$$

$$\sum s_i s_{i+a} = 0 \ \forall a \in \mathbf{Z} \tag{4.4}$$

$$\sum_{i<j} s_{j+\alpha}s_i = 0 \tag{4.5}$$

$$\sum_{i<j} s_{j-i+a}s_i s_{i+b} = 0 \tag{4.6}$$

Each of these results holds save at finitely many bad primes p, and the particular set of bad primes will be identified in each case. To be precise, we will identify a finite superset of the bad primes. Experience indicates that, when a prime is in the described superset, the relevant result fails to hold.

Proof

(4.1) must clearly hold save when p divides $m + l - 1$.

(4.2) requires a little more work. We shall calculate the sum in two ways, and then exploit both pieces of information. We have

$$\sum s_i^2 = \sum (ls_{i-1} + ms_{i-2})^2 = (l^2 + m^2)\sum s_i^2 + 2ml\sum s_i s_{i-1},$$

and so

$$(l^2 + m^2 - 1)\sum s_i^2 + 2ml\sum s_i s_{i-1} = 0.$$

Now we seek a second equation relating $\sum s_i^2$ and $\sum s_i s_{i-1}$. We have

$$m^2\sum s_i^2 = \sum (s_{i+2} - ls_{i+1})^2 = (1 + l^2)\sum s_i^2 - 2l\sum s_i s_{i-1},$$

so that

$$(1 + l^2 - m^2)\sum s_i^2 - 2l\sum s_i s_{i-1} = 0.$$

We now have two linear equations relating $\sum s_i^2$ and $\sum s_i s_{i-1}$. The relevant determinant turns out to factor into four pleasant terms. The determinant is

$$2l(-m-1)(l-m+1)(l+m-1).$$

Thus, providing p fails to divide each these four factors, both $\sum s_i^2$ and $\sum s_i s_{i-1}$ must vanish. The fourth condition co-incides with the condition for (4.1) to hold.

Now for equation (4.4). Assuming that we impose the conditions necessary for (4.2) and (4.3) to hold, then equation (4.4) follows automatically. This is because

$$\sum s_i s_{i+2} = \sum s_i(ls_{i+1} + ms_i) = l\sum s_i s_{i+1} + m\sum s_i^2.$$

Thus $\sum s_i s_{i+2}$ vanishes. Now equation (4.4) follows by the recurrence relation.

We move to consider equation (4.5). Now

$$l\sum_{i<j} s_{j+a}s_i = \sum_{i<j} s_{j+a}(s_{i+1} - ms_{i-1}) = \sum_{i<j} s_{j+a}s_{i+1} - m\sum_{i<j} s_{j+a}s_{i-1}.$$

Let us consider the first sum in the final expression. We have

$$\sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_{i+1} = \sum_{j=0}^{k-1}\sum_{i=1}^{j} s_{j+a}s_i = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_i - \sum_{j=0}^{k-1} s_{j+a}s_0 + \sum_{j=0}^{k-1} s_{j+a}s_j.$$

Providing we impose the conditions on l and m to make equations (4.1) and (4.4) hold the last two terms vanish, and we deduce that

$$\sum_{i<j} s_{j+a}s_i = \sum_{i<j} s_{j+a}s_{i+1}.$$

Now we proceed to discuss the second term. We have

$$\sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_{i-1} = \sum_{j=0}^{k-1}\sum_{i=-1}^{j-2} s_{j+a}s_i = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_i + \sum_{j=0}^{k-1} s_{j+a}s_{-1} - \sum_{j=0}^{k-1} s_{j+a}s_{j-1}.$$

The last two expressions vanish by (4.1) and (4.4), and so

$$\sum_{i<j} s_{j+a}s_{i-1} = \sum_{i<j} s_{j+a}s_i.$$

Now we put all this information together to obtain

$$(l+m-1)\sum_{i<j} s_{j+a}s_i = 0.$$

Now we move to equation (4.6). We put

$$d_{a,b} = \sum_{i<j} s_{j-i+a}s_is_{i+b}.$$

Now

$$s_{j-i+a} = ls_{j-i+a-1} + ms_{j-i+a-2},$$

so that

$$d_{a,b} = ld_{a-1,b} + md_{a-2,b}$$

for all integers a and b. Similarly we have

$$d_{a,b} = ld_{a,b-1} + md_{a,b-2}.$$

Now, for all α and β we have

$$s_{\alpha+\beta} = s_\alpha s_{\beta+1} + (s_{\alpha+1} - ls_\alpha)s_\beta.$$

Put $\alpha = j - i + a$ and $\beta = i + b$. Multiply both sides by s_i and integrate over the range $0 \leq i < j < k$. Thanks to equation (4.5) we obtain

$$d_{a,b+1} + d_{a+1,b} - ld_{a,b} = 0.$$

We seek yet another system of linear equations which will be satisfied by the quantities $d_{a,b}$. Now

$$d_{a,b} = \sum_{i<j} s_{j-i+a}(ls_{i-1} + ms_{i-2})s_{i+b} = l\sum_{i<j} s_{j-i+a}s_{i-1}s_{i+b} + m\sum_{i<j} s_{j-i+a}s_{i-2}s_{i+b}.$$

We now massage the two sums on the right of this equation. We have

$$\sum_{i<j} s_{j-i+a}s_{i-1}s_{i+b} = \sum_{j=0}^{k-1}\sum_{i=-1}^{j-2} s_{j-i+a-1}s_is_{i+b+1}$$

$$= \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j-i+a-1}s_is_{i+b+1} + \sum_{j=0}^{k-1} s_{j+a}s_{-1}s_b - \sum_{j=0}^{k-1} s_as_{j-1}s_{j+b}.$$

The last two sums vanish by (4.1) and (4.4). Therefore

$$\sum_{i<j} s_{j-i+a}s_{i-1}s_{i+b} = \sum_{i<j} s_{j-i+a-1}s_is_{i+b+1}.$$

Now the second sum of our main equation

$$\sum_{i<j} s_{j-i+a}s_{i-2}s_{i+b} = \sum_{j=0}^{k-1}\sum_{i=-2}^{j-3} s_{j-i+a-2}s_is_{i+b+2} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j-i+a-2}s_is_{i+b+2}$$

$$+\sum_{j=0}^{k-1} s_{j+a}s_{-2}s_b + \sum_{j=0}^{k-1} s_{j+a-1}s_{-1}s_{b+1} - \sum_{j=0}^{k-1} s_a s_{j-2}s_{j+b} - \sum_{j=0}^{k-1} s_{a-1}s_{j-1}s_{j+b+1}.$$

The last four sums vanish by (4.1) and (4.4). Therefore

$$\sum_{i<j} s_{j-i+a}s_{i-2}s_{i+b} = \sum_{i<j} s_{j-i+a-2}s_i s_{i+b+2}.$$

Thus

$$d_{a,b} = ld_{a-1,b+1} + md_{a-2,b+2}.$$

We now have four systems of linear equations amongst the quantities $d_{a,b}$, which are valid whenever the prime p fails to divide

$$2l(-m-1)(l-m+1)(l+m-1).$$

For small, specific values of l and m these systems of equations can be shown to force each $d_{a,b}$ to vanish. This is simply a matter of linear algebra, and the details follow. One must notice that $d_{a,b}$ and $c_{a,b}$ satisfy exactly the same equations except the diagonal one. Indeed, it is plain that the diagonal equation is also the same. To see this it is enough to exchange a and b in the definition of $d_{a,b}$ then the direction of the diagonal becomes the same. Thus $d_{a,b}$ vanishes under the same conditions as $c_{a,b}$ and we refer to the AXIOM code (see cabstep2 code) and its result for the values of l and m in table 3.1.

Corollary 4.1.2 *For all $a, b, c \in \mathbf{Z}$ we have*

$$\sum_{i<j} s_{j-i+a}s_{i+c}s_{i+b} = 0. \tag{4.7}$$

The next quantities which we shall work on are

$$\sum_{j=0}^{k-1} s_{j+a}s_{j+b}s_{-j+c}s_j.$$

Lemma 4.1.3 *For all $a, b, c \in \mathbf{Z}$ we have*

$$\sum_{j=0}^{k-1} s_{j+a}s_{j+b}s_{-j+c}s_j = 0. \tag{4.8}$$

Proof Let $X_{a,b,c} = \sum_j s_{j+a}s_{j+b}s_{-j+c}s_j$. Let

$$Y_{a,b} = (X_{a,b,0}, X_{a,b,1}, X_{a,b,2}, \ldots),$$

$$Y^+_{a,b} = (X_{a,b,1}, X_{a,b,2}, X_{a,b,3}, \ldots)$$

and

$$Y^-_{a,b} = (X_{a,b,-1}, X_{a,b,0}, X_{a,b,1}, \ldots).$$

We have various equations:

$$s_t = ls_{t-1} + ms_{t-2}, \tag{4.9}$$

$$s_{\alpha+\beta} = s_\alpha s_{\beta+1} + (s_{\alpha+1} - ls_\alpha)s_\beta. \tag{4.10}$$

The first equation is the definition of the 2–step general Fibonacci recurrence and the second is the same as (3.32). The quantities $X_{a,b,c}$ satisfy the following equations.

$$X_{a,b,c} = lX_{a-1,b,c} + mX_{a-2,b,c}, \tag{4.11}$$

$$X_{a,b,c} = lX_{a,b-1,c} + mX_{a,b-2,c}, \tag{4.12}$$

$$X_{a,b,c} = lX_{a,b,c-1} + mX_{a,b,c-2} \tag{4.13}$$

$$X_{a,b,c} = lX_{a+1,b+1,c-1} + mX_{a+2,b+2,c-2} \tag{4.14}$$

Put $\alpha = j + a$ and $\beta = -j + c$ in (4.10) to get:

$$s_{a+c} = s_{j+a}s_{-j+c} + (s_{j+a+1} - ls_{j+a})s_{-j+c-1}.$$

Multiply both sides by $s_j s_{j+b}$ and sum over the fundamental range to obtain

$$0 = X_{a,b,c} + X_{a+1,b,c-1} - lX_{a,b,c-1}. \tag{4.15}$$

Now we obtain equations for the quantities $Y_{a,b}$.

$$Y_{a,b} = lY_{a-1,b} + mY_{a-2,b}, \tag{4.16}$$

$$Y_{a,b} = lY_{a,b-1} + mY_{a,b-2}, \tag{4.17}$$

$$Y_{a,b} = lY^{-}_{a,b} + mY^{--}_{a,b}, \tag{4.18}$$

$$Y_{a,b} = lY^{-}_{a+1,b+1} + mY^{--}_{a+2,b+2}, \tag{4.19}$$

$$0 = Y_{a,b} + Y^{-}_{a+1,b} - lY^{-}_{a,b} \tag{4.20}$$

or

$$Y_{a+1,b} = -Y^{+}_{a,b} + lY_{a,b}. \tag{4.21}$$

Now

$$s_{\alpha+\beta} = s_\alpha s_\beta + (s_{\alpha+1} - ls_\alpha)s_{\beta-1}.$$

Put $\beta = -j + c$, $\alpha = j + b$, multiply through by $s_{j+a}s_j$, and sum over the fundamental range to get:

$$0 = X_{a,b,c} + (X_{a,b+1,c-1} - lX_{a,b,c-1}).$$

Thus

$$0 = Y_{a,b} + (Y^{-}_{a,b+1} - lY^{-}_{a,b})$$

so that

$$Y^{-}_{a,b+1} = -Y_{a,b} + lY^{-}_{a,b}$$

or

$$Y_{a,b+1} = -Y^{+}_{a,b} + lY_{a,b}. \tag{4.22}$$

Let us represent

$$Y_{r,s} = (X_{r,s,0}, X_{r,s,1}, \ldots)$$

via its transpose, and truncate the vertical sequence to have length 2. Thus we replace $Y_{r,s}$ by $Y^T_{r,s}$ truncated.

Rotation is then accomplished by left multiplication by the matrix

$$c = \begin{pmatrix} 0 & 1 \\ l & m \end{pmatrix}.$$

(4.21) gives

$$Y_{a+1,b} = (l - c)Y_{a,b}$$

and (4.22) gives

$$Y_{a,b+1} = (l - c)Y_{a,b}.$$

Let $A = l - c$. Put

$$\bar{a} = Y_{\alpha,\beta} = (\bar{a}_1, \bar{a}_2)^T$$

and consider the 3 by 3 matrix:

$$\begin{pmatrix} A^2\bar{a} & A^3\bar{a} & A^4\bar{a} \\ A\bar{a} & A^2\bar{a} & A^3\bar{a} \\ \bar{a} & A\bar{a} & A^2\bar{a} \end{pmatrix}.$$

The procedure is as follows. We use the matrix A to generate the entries of the 3 by 3 table shown above using the horizontal and vertical $Y-$ equations. We deploy equation (4.19) to give ourselves 'diagonal' equations. We then take these equations, interpret them as being equations over the integers, and determine that, save for bad primes, the system is of full rank. The calculation was made on a computer using the Computer Algebra system AXIOM [35], (see xabcstep2 code). We give the results in table 4.1 shows for which primes $Y_{a,b} \neq 0$ for $1 \leq l, m \leq 10$. Thus save for the primes given table 4.1, we may deduce that $X_{a,b,c} = 0$ for all integers a, b and c.

l,m	primes	l,m	primes	l,m	primes
1,1	3	4,5	2,23,61	7,9	5,7,16651
1,2	7	4,6	2,5,2729	7,10	3,7,206641
1,3	17,43	4,7	2,3,7,2029	8,1	2,19073
1,4	3,461	4,8	2,241,3121	8,2	2,3,7,73
1,5	5,631	4,9	2,43,10937	8,3	2,3,5,3259
1,6	151,311	4,10	2,3,13,35677	8,4	2,11,18371
1,7	3,1459	5,1	5,1151	8,5	2,3,41,59
1,8	262981	5,2	3,5,577	8,6	2,3,239
1,9	7,1553	5,3	3,5,7	8,7	2,22639
1,10	3,5,113,197	5,4	5,5003	8,8	2,3,5,283
2,1	2,31	5,5	3,5,29	8,9	2,3,7,31
2,2	2,3,11	5,6	3,5,7,13	8,10	2,996649
2,3	2,3,19	5,7	5,13,107,113	9,1	3,5,1471
2,4	2,5,1229	5,8	3,5,61261	9,2	3,690689
2,5	2,3,1583	5,9	3,5,171923	9,3	3,790291
2,6	2,3,7,1531	5,10	5,7,76249	9,4	3,8069
2,7	2,33287	6,1	2,3,13,277	9,5	3,19,21341
2,8	2,3,47,167	6,2	2,3,29429	9,6	3,5,7,23,41
2,9	2,3,5,15121	6,3	2,3,2087	9,7	3,349
2,10	2,11,243239	6,4	2,3,277	9,8	3,83,953
3,1	3,17	6,5	2,3,5,1171	9,9	3,17,1153
3,2	3,463	6,6	2,3,11,409	9,10	3,151,271
3,3	3,5,79	6,7	2,3,41,127	10,1	2,3,5,37,197
3,4	3,31,37	6,8	2,3,317321	10,2	2,5,1471693
3,5	3,7,3467	6,9	2,3,23,103,223	10,3	2,5,31,14831
3,6	3,99289	6,10	2,3,5,223,853	10,4	2,3,5,13,14621
3,7	3,103399	7,1	3,7,991	10,5	2,5,7,29,3121
3,8	3,5,31,5227	7,2	7,89,167	10,6	2,5,147409
3,9	3,19,29,3371	7,3	7,73,1193	10,7	2,3,5,223
3,10	3,277,4639	7,4	3,5,7,131	10,8	2,5,4391
4,1	2,3,5	7,5	7,13,107	10,9	2,5,29,2251
4,2	2,191	7,6	7,23,1049	10,10	2,3,5,13,19
4,3	2,887	7,7	3,7,13,67		
4,4	2,3,7,17	7,8	7,13,103		

Table 4.1:

Another family of sums which we must study are

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-1}s_is_{i+c}.$$

Lemma 4.1.4 *For all integers a, b and c we have*

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-1}s_is_{i+c} = 0. \tag{4.23}$$

Proof Fix c for the purpose of the proof. Let

$$\theta_{a,b} = \sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-1}s_is_{i+c}.$$

Clearly

$$\theta_{a,b} = l\theta_{a-1,b} + m\theta_{a-2,b},$$

$$\theta_{a,b} = l\theta_{a,b-1} + m\theta_{a,b-2}.$$

Consider

$$s_{\alpha+\beta} = s_\alpha s_{\beta+1} + (s_{\alpha+1} - ls_\alpha)s_\beta.$$

Put $\alpha = -j + a$ and $\beta = j + b$ to obtain

$$s_{a+b} = s_{-j+a}s_{j+b+1} + (s_{-j+a+1} - ls_{-j+a})s_{j+b}.$$

Multiply both sides by $s_{j-i-1}s_is_{i+c}$ and take the double sum over the range $0 \leq i < j < k$. Thanks to the equation (4.6), we get

$$0 = \theta_{a,b+1} + \theta_{a+1,b} - l\theta_{a,b}.$$

Also we have

$$s_{j-i-1} = ls_{j-i-2} + ms_{j-i-3},$$

so that

$$\theta_{a,b} = l \sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-2} s_i s_{i+c} + m \sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-3} s_i s_{i+c}.$$

Let us examine the first of these two sums. We see that

$$\sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-2} s_i s_{i+c} = \sum_{j=-1}^{k-2} \sum_{i=0}^{j} s_{-j+a-1} s_{j+b+1} s_{j-i-1} s_i s_{i+c}$$

$$= \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{-j+a-1} s_{j+b+1} s_{j-i-1} s_i s_{i+c} - \sum_{i=0}^{k-1} s_{-k+a} s_{k+b} s_{k-i-2} s_i s_{i+c}$$

$$+ \sum_{j=0}^{k-1} s_{-j+a-1} s_{j+b+1} s_{-1} s_j s_{j+c}.$$

By using (3.6.2) and (4.1.3), we have

$$\sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-2} s_i s_{i+c} = \sum_{i<j} s_{-j+a-1} s_{j+b+1} s_{j-i-1} s_i s_{i+c}.$$

Now the second sum of our main equation:

$$\sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-3} s_i s_{i+c} = \sum_{j=0}^{k-3} \sum_{i=0}^{j+1} s_{-j+a-2} s_{j+b+2} s_{j-i-1} s_i s_{i+c}$$

$$= \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{-j+a-2} s_{j+b+2} s_{j-i-1} s_i s_{i+c} - \sum_{i=0}^{k-1} s_{-k+a} s_{k+b} s_{k-i-3} s_i s_{i+c}$$

$$- \sum_{i=0}^{k} s_{-k+a-1} s_{k+b+1} s_{k-i-2} s_i s_{i+c} + \sum_{j=0}^{k-1} s_{-j+a-2} s_{j+b+2} s_j s_{-1} s_{j+c}$$

$$+ \sum_{j=0}^{k-1} s_{-j+a-2} s_{j+b+2} s_{j+1} s_{-2} s_{j+c+1}.$$

Again we use (3.6.2) and (4.1.3) to give

$$\sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-3} s_i s_{i+c} = \sum_{i<j} s_{j+a-2} s_{j+b+2} s_{j-i-1} s_i s_{i+c}.$$

Thus

$$\theta_{a,b} = l\theta_{a-1,b+1} + m\theta_{a-2,b+2}.$$

These equations are enough to force each $\theta_{a,b}$ to vanish except at finitely many bad primes. Notice that $\theta_{a,b}$ satisfies exactly the same *template equations* as the quantities of (4.6) and must vanish for the same reason.

Corollary 4.1.5 *For all integers a, b, c, d and e we have*

$$\sum_{i<j} s_{-j+a} s_{j+b} s_{j-i-d} s_{i+e} s_{i+c} = 0. \tag{4.24}$$

Our main ambition is to show that $d_k = \sum_{i=1}^{15} d_i = 0$. Now this is simply the upshot of (3.6.2), (4.1.3) and (4.1.5), as d_k is a linear combination of the quantities of those. Moreover, by the same argument, $d_{k+1} = 0$.

4.2 General 3–Step Recurrences

We now press on to the three step case. The basic recurrence is

$$s_i = l s_{i-1} + m s_{i-2} + n s_{i-3}.$$

We will have need of an auxiliary sequence (t_i) defined by

$$t_i = s_{i+1} - l s_i.$$

The three step class 3 case is the most difficult we shall work with in this thesis. We work with usual group H and, in the standard notation.

The question arises as to whether or not it is legitimate to work in H rather than in the full free nilpotent group of class 3 on 3 generators. We performed a calculation which justified this in the 3–step Fibonacci case (chapter 3). Exactly the same method *mutatis mutandis* will, we believe, work for general coefficients. We have not proved this, but claim that, if we are challenged with specific coefficients, then the routine calculation can be made. Unfortunately we do not have a general argument. The point is that, if the relevant intersection is trivial, then it is easy, for specified coefficients, to verify this. Thus the question is really reduced to 'is the intersection trivial?'. We have gathered strong evidence (for small l,m,n) that the intersection is indeed trivial (with the automorphism built from the coefficients) by examining a large number of examples. We find that, for specific primes p which happen to be 'good primes' for the relevant coefficients, the corresponding intersection in the exponent p group is trivial, where we assume that $p > 5$. (See the cayley code). This is not a proof, but it is persuasive. We are, after all, not claiming a general proof, but rather a general method.

We must show that all d_k, d_{k+1} and d_{k+2} vanish.

This time d_k splits into 19 relatively easy sums. They are

$$d_1 = \binom{n}{2}\sum s_{k-i-1}\binom{t_i}{2}s_i,$$

$$d_2 = \binom{n}{2}\sum s_{k-i-1}t_ic_i,$$

$$d_3 = \binom{n}{3}\sum s_{k-i-1}t_i^2s_i,$$

$$d_4 = \binom{m}{2}\sum s_{k-i-1}\binom{t_{i+1}}{2}s_{i+1},$$

$$d_5 = \binom{m}{2}\sum s_{k-i-1}t_{i+1}c_{i+1},$$

$$d_6 = \binom{m}{3}\sum s_{k-i-1}t_{i+1}^2s_{i+1},$$

$$d_7 = mn\sum s_{k-i-1}t_{i+1}c_i,$$

$$d_8 = m\binom{n}{2}\sum s_{k-i-1}t_it_{i+1}s_i,$$

$$d_9 = n\sum s_{k-i-1}\binom{mt_{i+1}}{2}s_i,$$

$$d_{10} = \binom{l}{2}\sum s_{k-i-1}\binom{t_{i+2}}{2}s_{i+2},$$

$$d_{11} = \binom{l}{2}\sum s_{k-i-1}t_{i+1}c_{i+2},$$

$$d_{12} = \binom{l}{3}\sum s_{k-i-1}t_{i+2}^2s_{i+2},$$

$$d_{13} = ln\sum s_{k-i-1}t_{i+2}c_i,$$

$$d_{14} = lm\sum s_{k-i-1}t_{i+2}c_{i+1},$$

$$d_{15} = l\binom{n}{2}\sum s_{k-i-1}t_{i+2}t_is_i,$$

$$d_{16} = l\binom{m}{2}\sum s_{k-i-1}t_{i+2}t_{i+1}s_{i+1},$$

$$d_{17} = lmn\sum s_{k-i-1}t_{i+2}t_{i+1}s_i,$$

$$d_{18} = n\sum s_{k-i-1}\binom{lt_{i+2}}{2}s_i$$

and

$$d_{19} = m\sum s_{k-i-1}\binom{lt_{i+2}}{2}s_{i+1}.$$

We now simplify each of these sums. Now for d_1. Ignore the "bad prime" 2. We have d_1 is a linear combination of

$$\sum s_{k-i-1}s_{i+1}^2s_i,$$

$$\sum s_{k-i-1}s_{i+1}s_i^2,$$

$$\sum s_{k-i-1}s_i^3,$$

$$\sum s_{k-i-1}s_{i+1}s_i$$

and

$$\sum s_{k-i-1}s_i^2.$$

Now for d_2. Ignore the "bad prime" 2. We have d_2 is a linear combination of

$$\sum_{i<j} s_{k-j-1}t_js_{j-i-1}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_js_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_js_{j-i-1}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}t_js_{j-i-1}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}t_js_{j-i-1}s_{i+3}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_j s_{j-i-1}s_{i+3}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_j s_{j-i-1}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_j s_{j-i-1}s_{i+3}s_{i+2}$$

and

$$\sum_{i<j} s_{k-j-1}t_j s_{j-i-1}s_{i+2}^2.$$

Now for d_3. Ignore the "bad primes" 2 and 3. We have d_3 is a linear combination of

$$\sum s_{k-i-1}s_{i+1}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+1}s_i^2$$

and

$$\sum s_{k-i-1}s_i^3.$$

Now for d_4. Ignore the "bad prime" 2. We have d_4 is a linear combination of

$$\sum s_{k-i-1}s_{i+2}^2 s_{i+1},$$

$$\sum s_{k-i-1}s_{i+2}s_{i+1}^2,$$

$$\sum s_{k-i-1}s_{i+1}^3,$$

$$\sum s_{k-i-1}s_{i+2}s_{i+1}$$

and

$$\sum s_{k-i-1}s_{i+1}^2.$$

Now for d_5. Ignore the "bad prime" 2. We have d_5 is a linear combination of

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+3}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+3}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+3}s_{i+2}$$

and

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i}s_{i+2}^2.$$

Now for d_6. Ignore the "bad primes" 2 and 3. We have d_6 is a linear combination of

$$\sum s_{k-i-1}s_{i+2}^2s_{i+1},$$

$$\sum s_{k-i-1}s_{i+2}s_{i+1}^2$$

and

$$\sum s_{k-i-1}s_{i+1}^3.$$

Now for d_7. Ignore the "bad prime" 2. We have d_7 is a linear combination of

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+3}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+3}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+3}s_{i+2}$$

and

$$\sum_{i<j} s_{k-j-1}t_{j+1}s_{j-i-1}s_{i+2}^2.$$

Now for d_8. Ignore the "bad prime" 2. We have d_8 is a linear combination of

$$\sum s_{k-i-1}s_{i+1}s_{i+2}s_i,$$

$$\sum s_{k-i-1}s_{i+1}^2 s_i,$$

$$\sum s_{k-i-1}s_i^2 s_{i+2}$$

and

$$\sum s_{k-i-1}s_i^2 s_{i+1}.$$

Now for d_9. Ignore the "bad prime" 2. We have d_9 is a linear combination of

$$\sum s_{k-i-1}s_{i+2}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+1}s_{i+2}s_i,$$

$$\sum s_{k-i-1}s_{i+1}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+2}s_i$$

and

$$\sum s_{k-i-1}s_i^2.$$

Now for d_{10}. Ignore the "bad prime" 2. We have d_{10} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}^2 s_{i+2},$$

$$\sum s_{k-i-1}s_{i+3}s_{i+2}^2,$$

$$\sum s_{k-i-1}s_{i+2}^3,$$

$$\sum s_{k-i-1}s_{i+3}s_{i+2}$$

and

$$\sum s_{k-i-1}s_{i+2}^2.$$

Now for d_{11}. Ignore the "bad prime" 2. We have d_{11} is a linear combination of

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+3}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+3}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+3}s_{i+2}$$

and

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i+1}s_{i+2}^2.$$

Now for d_{12}. Ignore the "bad primes" 2 and 3. We have d_{12} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}^2 s_{i+2},$$

$$\sum s_{k-i-1}s_{i+2}^2 s_{i+3}$$

and

$$\sum s_{k-i-1}s_{i+2}^3.$$

Now for d_{13}. Ignore the "bad prime" 2. We have d_{13} is a linear combination of

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+3}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+3}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+3}s_{i+2}$$

and

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i-1}s_{i+2}^2.$$

Now for d_{14}. Ignore the "bad prime" 2. We have d_{14} is a linear combination of

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+2}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+1}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_i^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+2}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+1}^2,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+3}s_i,$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+3}s_{i+1},$$

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+3}s_{i+2}$$

and

$$\sum_{i<j} s_{k-j-1}t_{j+2}s_{j-i}s_{i+2}^2.$$

Now for d_{15}. Ignore the "bad prime" 2. We have d_{15} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}s_{i+1}s_i,$$

$$\sum s_{k-i-1}s_{i+3}s_i^2,$$

$$\sum s_{k-i-1}s_{i+2}s_{i+1}s_i$$

and

$$\sum s_{k-i-1}s_{i+2}s_i^2.$$

Now for d_{16}. Ignore the "bad prime" 2. We have d_{16} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}s_{i+2}s_{i+1},$$

$$\sum s_{k-i-1}s_{i+3}s_{i+1}^2,$$

$$\sum s_{k-i-1}s_{i+2}^2s_{i+1}$$

and

$$\sum s_{k-i-1}s_{i+2}s_{i+1}^2.$$

Now for d_{17}.We have d_{17} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}s_{i+2}s_i,$$

$$\sum s_{k-i-1}s_{i+3}s_{i+1}s_i,$$

$$\sum s_{k-i-1}s_{i+2}^2 s_i$$

and

$$\sum s_{k-i-1}s_{i+2}s_{i+1}s_i.$$

Now for d_{18}. Ignore the "bad prime" 2. We have d_{18} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+3}s_{i+2}s_i,$$

$$\sum s_{k-i-1}s_{i+2}^2 s_i,$$

$$\sum s_{k-i-1}s_{i+3}s_i$$

and

$$\sum s_{k-i-1}s_{i+2}s_i.$$

Now for d_{19}. Ignore the "bad prime" 2. We have d_{19} is a linear combination of

$$\sum s_{k-i-1}s_{i+3}^2 s_{i+1},$$

$$\sum s_{k-i-1}s_{i+3}s_{i+2}s_{i+1},$$

$$\sum s_{k-i-1}s_{i+2}^2 s_{i+1},$$

$$\sum s_{k-i-1}s_{i+3}s_{i+1}$$

and

$$\sum s_{k-i-1}s_{i+2}s_{i+1}.$$

We now address the question as to why all these sums must vanish. In some cases we will have recourse to linear algebra results obtained using AXIOM. These will appear in the next chapter.

We shall assume that the initial data for the sequence is

$$(s_0, s_1, s_2) = (0, 0, 1)$$

though this fact is only used sparingly. As a consequence, many of the subsequent results are independent of the initial data, though we shall not need this. Before embarking on the general theory, we will first dispose of a few routine sums,

Lemma 4.2.1

$$\sum s_i = 0 \tag{4.25}$$

$$\sum s_i s_{i-1} = 0 \tag{4.26}$$

$$\sum s_i^2 = 0 \tag{4.27}$$

$$\sum s_i s_{i+a} = 0 \ \forall a \in \mathbf{Z} \tag{4.28}$$

$$\sum_{i<j} s_{j+a} s_i = 0 \tag{4.29}$$

$$\sum_{i<j} s_{j-i+a} s_i s_{i+b} = 0 \tag{4.30}$$

Each of these results holds save at finitely many bad primes p, and the particular set of bad primes will be identified in each case. To be precise, we will identify a finite superset of the bad primes. Experience indicates that when a prime is in the described superset, the relevant result fails to hold.

Proof

(4.25) must clearly hold save when p divides $n + m + l - 1$.

(4.26) requires a little more work. We shall calculate the sum in two ways, and then exploit both pieces of information. We have

$$\sum s_i^2 = \sum (n s_{i-3} + m s_{i-2} + l s_{i-1})^2$$

$$= (n^2 + m^2 + l^2)\sum s_i^2 + 2nm\sum s_{i-3}s_{i-2} + 2nl\sum s_{i-3}s_{i-1} + 2ml\sum s_{i-2}s_{i-1}$$

$$= (n^2+m^2+l^2)\sum s_i^2+2nm\sum s_{i-3}s_{i-2}+2l\sum(s_i-ls_{i-1}-ms_{i-2})s_{i-1}+2ml\sum s_{i-2}s_{i-1}$$

$$= (n^2 + m^2 + l^2)\sum s_i^2 + 2nm\sum s_{i-3}s_{i-2} + 2l\sum s_is_{i-1} - 2l^2\sum s_{i-1}^2 - 2lm\sum s_{i-2}s_{i-1}$$

$$+2ml\sum s_{i-2}s_{i-1},$$

so that

$$\sum s_i^2 = (n^2 + m^2 - l^2)\sum s_i^2 + 2(l + mn)\sum s_is_{i-1},$$

and so

$$(1 - n^2 - m^2 + l^2)\sum s_i^2 - 2(l + mn)\sum s_is_{i-1} = 0.$$

Now we seek a second equation relating $\sum s_i^2$ and $\sum s_is_{i-1}$. We have

$$\sum s_i^2 = \sum(ns_{i-3} + ms_{i-2} + ls_{i-1})^2$$

$$= (n^2 + m^2 + l^2)\sum s_i^2 + 2nm\sum s_{i-3}s_{i-2} + 2nl\sum s_{i-3}s_{i-1} + 2ml\sum s_{i-2}s_{i-1}$$

$$= (n^2 + m^2 + l^2)\sum s_i^2 + 2m(n + l)\sum s_is_{i-1} + 2nl\sum s_{i-3}(ns_{i-4} + ms_{i-3} + ls_{i-2})$$

$$= (n^2 + m^2 + l^2)\sum s_i^2 + 2m(n + l)\sum s_is_{i-1} + 2n^2l\sum s_{i-3}s_{i-4} + 2nml\sum s_{i-3}^2$$

$$+2nl^2\sum s_{i-3}s_{i-2}$$

$$= (n^2 + m^2 + l^2 + 2nml)\sum s_i^2 + (2nm + 2ml + 2n^2l + 2nl^2)\sum s_is_{i-1},$$

and so

$$(1 - n^2 - m^2 - l^2 - 2nml)\sum s_i^2 - 2(nm + ml + n^2l + nl^2)\sum s_is_{i-1} = 0.$$

We now have two linear equations relating $\sum s_i^2$ and $\sum s_is_{i-1}$. The relevant determinant turns out to factor into four pleasant terms. The determinant is

$$2l(n - m + l + 1)(n + m + l - 1)(n^2 - ln - m - 1).$$

Thus, providing p fails to divide each these four factors, then both $\sum s_i^2$ and $\sum s_i s_{i-1}$ must vanish. The third condition co-incides with the condition for (4.25) to hold. The calculation of the determinant was performed with the assistance of AXIOM.

Now for equation (4.28). Assuming that we impose the conditions necessary for (4.26) and (4.27) to hold, then equation (4.28) follows automatically. This is because

$$\sum s_i s_{i+2} = \sum s_i(n s_{i-1} + m s_i + l s_{i+1}) = n \sum s_i s_{i-1} + m \sum s_i^2 + l \sum s_i s_{i+1}.$$

Thus $\sum s_i s_{i+2}$ vanishes. Now equation (4.28) follows by the recurrence relation.

We move to consider equation (4.29). Now

$$l \sum_{i<j} s_{j+a} s_i = \sum_{i<j} s_{j+a}(s_{i+1} - n s_{i-2} - m s_{i-1})$$

$$= \sum_{i<j} s_{j+a} s_{i+1} - n \sum_{i<j} s_{j+a} s_{i-2} - m \sum_{i<j} s_{j+a} s_{i-1}.$$

Let us consider the first sum in the final expression. We have

$$\sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{j+a} s_{i+1} = \sum_{j=0}^{k-1} \sum_{i=1}^{j} s_{j+a} s_i = \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{j+a} s_i - \sum_{j=0}^{k-1} s_{j+a} s_0 + \sum_{j=0}^{k-1} s_{j+a} s_j.$$

Providing we impose the conditions on l, m and n to make equations (4.25) and (4.28) hold the last two terms vanish, and we deduce that

$$\sum_{i<j} s_{j+a} s_i = \sum_{i<j} s_{j+a} s_{i+1}.$$

Now we proceed to discuss the second term. We have

$$\sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{j+a} s_{i-2} = \sum_{j=0}^{k-1} \sum_{j=-2}^{j-3} s_{j+a} s_i$$

$$= \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{j+a} s_i + \sum_{j=0}^{k-1} s_{j+a} s_{-2} + \sum_{j=0}^{k-1} s_{j+a} s_{-1} - \sum_{j=0}^{k-1} s_{j+a} s_{j-2} - \sum_{j=0}^{k-1} s_{j+a} s_{j-1}.$$

The last four sums vanish when equations (4.25) and (4.28) hold; so we may deduce that

$$\sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_{i-2} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_i.$$

In time honoured fashion, we now tackle the third sum on the right. We have

$$\sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_{i-1} = \sum_{j=0}^{k-1}\sum_{i=-1}^{j-2} s_{j+a}s_i = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{j+a}s_i + \sum_{j=0}^{k-1} s_{j+a}s_{-1} - \sum_{j=0}^{k-1} s_{j+a}s_{j-1}$$

The last two expressions vanish by (4.25) and (4.28) and so

$$\sum_{i<j} s_{j+a}s_{i-1} = \sum_{i<j} s_{j+a}s_i.$$

Now we put all this information together to obtain

$$(n+m+l-1)\sum_{i<j} s_{j+a}s_i = 0.$$

Now we move to equation (4.30). We put

$$d_{a,b} = \sum_{i<j} s_{j-i+a}s_i s_{i+b}.$$

Now

$$s_{j-i+a} = ns_{j-i+a-3} + ms_{j-i+a-2} + ls_{j-i+a-1},$$

so that

$$d_{a,b} = nd_{a-3,b} + md_{a-2,b} + ld_{a-1,b}$$

for all integers a and b. Similarly we have

$$d_{a,b} = nd_{a,b-3} + md_{a,b-2} + ld_{a,b-1}.$$

Now, for all α and β we have

$$s_{\alpha+\beta} = s_\alpha s_{\beta+2} + (s_{\alpha+1} - ls_\alpha)s_{\beta+1} + (s_{\alpha+2} - ls_{\alpha+1} - ms_\alpha)s_\beta.$$

Put $\alpha = j - i + a$ and $\beta = i + b$. Multiply both sides by s_i and integrate over the range $0 \leq i < j < k$. Thanks to equation (4.29), we obtain

$$d_{a,b+2} + d_{a+1,b+1} - l d_{a,b+1} + d_{a+2,b} - l d_{a+1,b} - m d_{a,b} = 0.$$

We seek yet another system of linear equations which will be satisfied by the quantities $d_{a,b}$. Now

$$d_{a,b} = \sum_{i<j} s_{j-i+a}(n s_{i-3} + m s_{i-2} + l s_{i-1}) s_{i+b}$$

$$= n \sum_{i<j} s_{j-i+a} s_{i-3} s_{i+b} + m \sum_{i<j} s_{j-i+a} s_{i-2} s_{i+b} + l \sum_{i<j} s_{j-i+a} s_{i-1} s_{i+b}.$$

We now massage the three sums on the right of this equation. We have

$$\sum_{i<j} s_{j-i+a} s_{i-3} s_{i+b} = \sum_{j=0}^{k-1} \sum_{i=-3}^{j-4} s_{j-i+a-3} s_i s_{i+b+3}$$

$$= \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} s_{j-i+a-3} s_i s_{i+b+3} + \sum_{j=0}^{k-1} s_{j+a} s_{-3} s_b + \sum_{j=0}^{k-1} s_{j+a-1} s_{-2} s_{b+1}$$

$$+ \sum_{j=0}^{k-1} s_{j+a-1} s_{-1} s_{b+2} - \sum_{j=0}^{k-1} s_a s_{j-3} s_{j+b} - \sum_{j=0}^{k-1} s_{a-1} s_{j-2} s_{j+b+1} - \sum_{j=0}^{k-1} s_{a-2} s_{j-1} s_{j+b+2}.$$

All sums bar the first in this expression vanish by (4.25) and (4.28). Thus

$$\sum_{i<j} s_{j-i+a} s_{i-3} s_{i+b} = \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} s_{j-i+a-3} s_i s_{i+b+3} = d_{a-3,b+3}.$$

A similar analysis applied to the second and third terms of our main equation yields that

$$d_{a,b} = n d_{a-3,b+3} + m d_{a-2,b+2} + l d_{a-1,b+1}.$$

We now have four systems of linear equations amongst the quantities $d_{a,b}$, which are valid whenever the prime p fails to divide

$$2l(n - m + l + 1)(n + m + l - 1)(n^2 - ln - m - 1).$$

For small, specific values of l, m and n these systems of equations can be shown to force each $d_{a,b}$ to vanish. This is simply a matter of linear algebra, and we do not give the details, as they are the same as in the proof of (4.6) except that the step number is different. Thus we refer to the AXIOM code (see cabstep3 code) and its result for the values of l, m and n in table 3.2.

Corollary 4.2.2 *For all $a, b, c \in \mathbf{Z}$ we have*

$$\sum_{i<j} s_{j-i+a} s_{i+c} s_{i+b} = 0. \tag{4.31}$$

The next family of sums which we must study are

$$\sum_{j=0}^{k-1} s_{j+a} s_{j+b} s_{-j+c} s_j.$$

Lemma 4.2.3 *For all integers a, b and c we have*

$$\sum_{j=0}^{k-1} s_{j+a} s_{j+b} s_{-j+c} s_j = 0. \tag{4.32}$$

Proof Let $X_{a,b,c} = \sum_j s_{j+a} s_{j+b} s_{-j+c} s_j$. Let

$$Y_{a,b} = (X_{a,b,0}, X_{a,b,1}, X_{a,b,2}, \ldots),$$

$$Y_{a,b}^{+} = (X_{a,b,1}, X_{a,b,2}, X_{a,b,3}, \ldots)$$

and

$$Y_{a,b}^{-} = (X_{a,b,-1}, X_{a,b,0}, X_{a,b,1}, \ldots)$$

We have various equations:

$$s_t = l s_{t-1} + m s_{t-2} + n s_{t-3}, \tag{4.33}$$

$$s_{\alpha+\beta} = s_\alpha s_{\beta+2} + (s_{\alpha+1} - l s_\alpha) s_{\beta+1} + (s_{\alpha+2} - l s_{\alpha+1} - m s_\alpha) s_\beta \ \forall \alpha, \beta. \tag{4.34}$$

The quantities $X_{a,b,c}$ satisfy the following equations.

$$X_{a,b,c} = nX_{a-3,b,c} + mX_{a-2,b,c} + lX_{a-1,b,c}, \tag{4.35}$$

$$X_{a,b,c} = nX_{a,b-3,c} + mX_{a,b-2,c} + lX_{a,b-1,c}, \tag{4.36}$$

$$X_{a,b,c} = nX_{a,b,c-3} + mX_{a,b,c-2} + lX_{a,b,c-1}, \tag{4.37}$$

$$X_{a,b,c} = nX_{a+3,b+3,c-3} + mX_{a+2,b+2,c-2} + lX_{a+1,b+1,c-1}. \tag{4.38}$$

Put $\alpha = j + a$ and $\beta = -j + c$ in (4.34) to get:

$$s_{a+c} = s_{j+a}s_{-j+c} + (s_{j+a+1} - ls_{j+a})s_{-j+c-1} + (s_{j+a+2} - ls_{j+a+1} - ms_{j+a})s_{-j+c-2}.$$

Multiply both sides by $s_j s_{j+b}$ and sum over the fundamental range to obtain

$$0 = X_{a,b,c} + (X_{a+1,b,c-1} - lX_{a,b,c-1}) + (X_{a+2,b,c-2} - lX_{a+1,b,c-2} - mX_{a,b,c-2}). \tag{4.39}$$

Now we obtain equations for the quantities $Y_{a,b}$.

$$Y_{a,b} = nY_{a-3,b} + mY_{a-2,b} + lY_{a-1,b}, \tag{4.40}$$

$$Y_{a,b} = nY_{a,b-3} + mY_{a,b-2} + lY_{a,b-1}, \tag{4.41}$$

$$Y_{a,b} = nY^{---}_{a,b} + mY^{--}_{a,b} + lY^{-}_{a,b}, \tag{4.42}$$

$$Y_{a,b} = nY^{---}_{a+3,b+3} + mY^{--}_{a+2,b+2} + lY^{-}_{a+1,b+1}, \tag{4.43}$$

$$0 = Y_{a,b} + (Y^{-}_{a+1,b} - lY^{-}_{a,b}) + (Y^{--}_{a+2,b} - lY^{--}_{a+1,b} - mY^{--}_{a,b}) \tag{4.44}$$

or

$$Y_{a+2,b} = -Y^{++}_{a,b} - Y^{+}_{a+1,b} + lY^{+}_{a,b} + lY_{a+1,b} + mY_{a,b}. \tag{4.45}$$

Consider

$$s_{\alpha+\beta} = s_\alpha s_\beta + (s_{\alpha+1} - ls_\alpha)s_{\beta-1} + (s_{\alpha+2} - ls_{\alpha+1} - ms_\alpha)s_{\beta-2}.$$

Put $\beta = -j + c,\ \alpha = j + b$, multiply through by $s_j s_{j+a}$ and sum over the fundamental range to get:

$$0 = X_{a,b,c} + (X_{a,b+1,c-1} - lX_{a,b,c-1}) + (X_{a,b+2,c-2} - lX_{a,b+1,c-2} - mX_{a,b,c-2})$$

and

$$0 = Y_{a,b} + (Y^{-}_{a,b+1} - lY^{-}_{a,b}) + (Y^{--}_{a,b+2} - lY^{--}_{a,b+1} - mY^{--}_{a,b})$$

so

$$Y^{--}_{a,b+2} = -Y_{a,b} - Y^{-}_{a,b+1} + lY^{-}_{a,b} + lY^{--}_{a,b+1} + mY^{--}_{a,b}$$

or

$$Y_{a,b+2} = -Y^{++}_{a,b} - Y^{+}_{a,b+1} + lY^{+}_{a,b} + lY_{a,b+1} + mY_{a,b}. \tag{4.46}$$

Let us represent

$$Y_{r,s} = (X_{r,s,0}, X_{r,s,1}, \ldots)$$

via its transpose, and truncate the vertical sequence to have length 3. Thus we replace $Y_{r,s}$ by $Y^T_{r,s}$ truncated.

Rotation is then accomplished by left multiplication by the matrix

$$c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ n & m & l \end{pmatrix}.$$

(4.45) gives

$$Y_{a+2,b} = (m + lc - c^2)Y_{a,b} + (l - c)Y_{a+1,b},$$

and (4.46) gives

$$Y_{a,b+2} = (m + lc - c^2)Y_{a,b} + (l - c)Y_{a,b+1}.$$

Let $A = m + lc - c^2$ and $B = l - c$. Put

$$\bar{a} = Y_{\alpha,\beta} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)^T,$$

$$\bar{b} = Y_{\alpha,\beta+1} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)^T,$$

$$\bar{c} = Y_{\alpha+1,\beta} = (\bar{c}_1, \bar{c}_2, \bar{c}_3)^T,$$

and

$$\bar{d} = Y_{\alpha+1,\beta+1} = (\bar{d}_1, \bar{d}_2, \bar{d}_3)^T$$

and then consider the 5 by 5 matrix:

$$\begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ A\bar{a} + B\bar{c} & A\bar{b} + B\bar{d} & * & * & * \\ \bar{c} & \bar{d} & A\bar{c} + B\bar{d} & * & * \\ \bar{a} & \bar{b} & A\bar{a} + B\bar{b} & A\bar{b} + BA\bar{a} + B^2\bar{b} & * \end{pmatrix}.$$

We use matrices A and B to determine the rows and columns of the given 5 by 5 array of vectors. We now deploy equation (4.43) in the four possible places within the array. Each of these new equations is a vector equation, and gives rise to 3 scalar equations. Now we have 12 linear equations in 12 unknowns. In time honoured fashion, we interpret these equations as being defined over the integers, and use a computer program to determine at which primes the system fails to be of full rank. Of course, when the system is of full rank, the unknowns are forced to vanish.

The row echelon form of the 12 by 12 matrix of these equations was calculated using the Computer Algebra system AXIOM [35]. (See xabcstep3 code).

We give the results in table 4.2. We specify which primes $Y_{a,b} \neq 0$ for $1 \leq l, m, n \leq 5$ are 'bad', and fail to give a matrix of full rank. Thus save for the primes given table 4.2, we may deduce that $X_{a,b,c} = 0$ for all integers a, b and c.

l,m,n	primes	l,m,n	primes
1,1,1	2,3	2,2,3	
1,1,2		2,2,4	2,3,5,7,43
1,1,3	2,3,83	2,2,5	2,3,5,17,12157
1,1,4	2,3,5,1283	2,3,1	2,3,5,101
1,1,5	2,3,5,173,1847	2,3,2	2,3,97,1997
1,2,1	2,3,47	2,3,3	2,3,7,13,7793
1,2,2	2,3,5	2,3,4	2,3,5,67,941
1,2,3	2,3,5,11,53,269	2,3,5	2,3,5,11,3167
1,2,4	2,3,37,541	2,4,1	
1,2,5	2,3,5,7,17,61	2,4,2	2,3,5,7
1,3,1		2,4,3	2,3,13,263,1321
1,3,2	2,3,5,137,1217	2,4,4	2,3,743,1451
1,3,3	2,3,11,673	2,4,5	2,3,5,151,21139
1,3,4	2,3,7,139,13709	2,5,1	2,3,7,11,31,311
1,3,5	2,3,5,2357	2,5,2	
1,4,1	2,3,5,47	2,5,3	2,3,47,67,6917
1,4,2		2,5,4	2,3,5,1193,11351
1,4,3	2,3,7,29,157	2,5,5	2,3,5,11,367
1,4,4	2,3,7,19,311	3,1,1	2,3,11
1,4,5	2,3,5,41,137,773	3,1,2	2,3,5,107,337
1,5,1	2,3,37,2521	3,1,3	2,3,3089,3779
1,5,2	2,3,7,1783,5647	3,1,4	2,3,7,41,131,919
1,5,3		3,1,5	2,3,5,23,1451
1,5,4	2,3,101,383,5743	3,2,1	2,3,5,53
1,5,5	2,3,5,7,29,197	3,2,2	2,3,5,157,167
2,1,1	2,3,263	3,2,3	2,3,5,7,211,4423
2,1,2	2,3,17,631	3,2,4	2,3,5,107,1559
2,1,3	2,3,5,19	3,2,5	2,3,5,7,13,79
2,1,4	2,3,53,89,421	3,3,1	2,3,5,499
2,1,5	2,3,5,7,13,41,47,109	3,3,2	2,3,7,11,457,521
2,2,1	2,3,5,23	3,3,3	2,3,59,577
2,2,2	2,3,5,167,211	3,3,4	

l,m,n	primes	l,m,n	primes
3,3,5	2,3,5,3517,3823	4,5,1	2,3,11,8237
3,4,1	2,3,7	4,5,2	2,3,5,521,28879
3,4,2	2,3,7,151,1439	4,5,3	2,3,11,17,41,9419
3,4,3	2,3,5,127,5861	4,5,4	2,3,127,197,1429
3,4,4	2,3,5,23,83,1061	4,5,5	2,3,5,13,29,67,8677
3,4,5	2,3,5,11,1873,19687	5,1,1	2,3,11,61,199
3,5,1		5,1,2	2,3,7,23,727,1109
3,5,2	2,3,37,191,17327	5,1,3	2,3,293,1523
3,5,3	2,3,5,5839,10453	5,1,4	2,3,22943,28087
3,5,4	2,3,11,37567,96443	5,1,5	2,3,5,37,43,59,3613
3,5,5	2,3,5,103,613	5,2,1	2,3,5,7,211
4,1,1	2,3,5,13,19	5,2,2	2,3,13,109,15383
4,1,2	2,3,3041,3137	5,2,3	2,3,7,13,71,101,421
4,1,3	2,3,5,7,17,4639	5,2,4	2,3,5,7,467,4603
4,1,4	2,3,701,101599	5,2,5	2,3,5,11,29,2467,8887
4,1,5	2,3,5,1049,12799	5,3,1	2,3,131,173
4,2,1	2,3,139,1297	5,3,2	2,3,5,67,347,10607
4,2,2	2,3,5,7,17,2083	5,3,3	2,3,5,11,97,2749
4,2,3	2,3,5,379,443	5,3,4	2,3,7,11,53,71,353,653
4,2,4	2,3,7,1277,6829	5,3,5	2,3,5,8273,19051
4,2,5	2,3,5,31,2017,2311	5,4,1	2,3,89,307,907
4,3,1	2,3,7,89,193	5,4,2	2,3,5,11,349,12941
4,3,2	2,3,43,97,227	5,4,3	2,3,5,11,11261,13999
4,3,3	2,3,5,7,83,86323	5,4,4	2,3,1579,126547
4,3,4	2,3,5,11,223,121229	5,4,5	2,3,5,7,13,131,409
4,3,5	2,3,5,7,11,17,101,167	5,5,1	2,3,5,13
4,4,1	2,3,17,139	5,5,2	2,3,11,48991,93979
4,4,2	2,3,19,23,367	5,5,3	2,3,53,139,167,233
4,4,3	2,3,5,2963,3299	5,5,4	2,3,5,13,37,89,1087
4,4,4	2,3,5,11,8941	5,5,5	2,3,5,7,3191,7549
4,4,5			

Table 4.2:

Another family of sums which we must study are

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-1}s_is_{i+c}.$$

Lemma 4.2.4 *For all integers a, b and c we have*

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-1}s_is_{i+c} = 0. \tag{4.47}$$

Proof Fix c for the purpose of the proof. Let

$$\theta_{a,b} = \sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-1}s_is_{i+c}.$$

Clearly

$$\theta_{a,b} = n\theta_{a-3,b} + m\theta_{a-2,b} + l\theta_{a-1,b},$$

$$\theta_{a,b} = n\theta_{a,b-3} + m\theta_{a,b-2} + l\theta_{a,b-1}.$$

Put $\alpha = -j + a$ and $\beta = j + b$ in (4.34) to obtain

$$s_{a+b} = s_{-j+a}s_{j+b+2} + (s_{-j+a+1} - ls_{-j+a})s_{j+b+1} + (s_{-j+a+2} - ls_{-j+a+1} - ms_{-j+a})s_{j+b}.$$

Multiply both sides by $s_{j-i-1}s_is_{i+c}$ and take the double sum over the range $0 \leq i < j < k$ to obtain, using (4.30),

$$0 = \theta_{a,b+2} + \theta_{a+1,b+1} - l\theta_{a,b+1} + \theta_{a+2,b} - l\theta_{a+1,b} - m\theta_{a,b}.$$

Also we have

$$s_{j-i-1} = ns_{j-i-4} + ms_{j-i-3} + ls_{j-i-2},$$

so that

$$\theta_{a,b} = n\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-4}s_is_{i+c} + m\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-3}s_is_{i+c}$$
$$+ l\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-2}s_is_{i+c}.$$

Let us examine the first of these three sums. We see that

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-4}s_is_{i+c} = \sum_{j=-3}^{k-4}\sum_{i=0}^{j+2} s_{-j+a-3}s_{j+b+3}s_is_{j-i-1}s_{i+c}$$

$$= \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{-j+a-3}s_{j+b+3}s_is_{j-i-1}s_{i+c} - \sum_{i=0}^{k-1} s_{-k+a}s_{k+b}s_is_{k-i-4}s_{i+c}$$

$$-\sum_{i=0}^{k} s_{-k+a-1}s_{k+b+1}s_is_{k-i-3}s_{i+c} - \sum_{i=0}^{k+1} s_{-k+a-2}s_{k+b+2}s_is_{k-i-2}s_{i+c}$$

$$+\sum_{i=0}^{1} s_{a-4}s_{b+2}s_is_{-i-2}s_{i+c} + \sum_{j=0}^{k-1} s_{-j+a-3}s_{j+b+3}s_js_{-1}s_{j+c}$$

$$+\sum_{j=0}^{k-1} s_{-j+a-3}s_{j+b+3}s_{j+1}s_{-2}s_{j+c+1} + \sum_{j=0}^{k-1} s_{-j+a-3}s_{j+b+3}s_{j+2}s_{-3}s_{j+c+2}.$$

By using (3.7.2) and (4.2.3), we get

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-4}s_is_{i+c} = \sum_{i<j} s_{-j+a-3}s_{j+b+3}s_{j-i-1}s_is_{i+c},$$

and similarly in the other two cases, to give

$$\theta_{a,b} = n\theta_{a-3,b+3} + m\theta_{a-2,b+2} + l\theta_{a-1,b+1}.$$

These equations are enough to force each $\theta_{a,b}$ to vanish except at finitely many bad primes. Notice that $\theta_{a,b}$ satisfies exactly the same *template equations* as (4.30) and must vanish for the same reason.

Corollary 4.2.5 *For all integers a, b, c, d and e we have*

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-d}s_{i+e}s_{i+c} = 0. \tag{4.48}$$

Our aim is to show that $d_k = \sum_{i=1}^{19} d_i = 0$. This is simply the upshot of (3.7.2), (4.2.3) and (4.2.5) as d_k is a linear combination of the quantities of those. By the same argument, $d_{k+1} = d_{k+2} = 0$, and we are done.

Conclusion

We have generalized the Wall-Vinson Theory of periodic sequences modulo a prime p to cover the $3-$step Fibonacci case. In particular, we have proved that non-trivial short loops must be geometric. Indeed, this result is true in the additive group of the field $GF(p^n)$, where the relevant manipulations are better performed. We also demonstrated that the corresponding theorem fails to hold for the Fibonacci $4-$step recurrence.

In another direction, we have generalized the Aydin-Smith theory of recurrences in finite $p-$groups. In particular, we have shown that, for the $3-$step Fibonacci recurrence and any finite $p-$group of exponent p and nilpotency class 3, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary $3-$step Fibonacci sequence in the field $GF(p)$. This involved doing the class 2 problem on the way. The corresponding theorem for class 4 is false.

We have addressed the question of recurrences with arbitrary coefficients. While we are not able to prove the corresponding theorem for either 2 or 3 step recurrences in general, we present a method for attacking individual questions. Given a specific recurrence, we believe that our method will prove a theorem of the form "For this particular recurrence, Wall's number behaves well in groups of exponent p having nilpotency class up to 3, except at finitely many bad primes". We have also presented experimental evidence that our method is likely to always produce a proof, though we can not guarantee this.

Some parts of this work show that complicated fourier sums are zero. We have given a mathematical proof when we deal with Fibonacci sequences and, in the work with general recurrences, we deploy Computer Algebra systems.

Our method involves machine assisted proofs and we have used the Computer Algebra systems CAYLEY and AXIOM (ex SCRATCHPAD) to verify our results. In particular, it was necessary to determine the ranks of large systems of linear equations, involving as many as 800 equations in 500 unknowns.

Work is also in progress considering the case when the group has nilpotency class 4. Although, this problem seems hard. Wall's number does sometimes increase when passing to the class 4 case. Some restriction on the prime numbers in question will be necessary to enable us to make the assertion that Wall's number does not increase.

Appendix

"The code called cabstep2"

```
)set message time on
)spool cabstep2.out
step:= 2
--input width of rectangle
w := 4
--input height of rectangle
h := 3
--program calculates number of equations generated
horiz:= max {0,h*(w-step)}
vert := max {0,w*(h-step)}
diag := max {0,(h-step)*(w-step)}
triang := max{0,(h-step+1)*(w-step+1)}
numeqs := horiz + vert + diag + triang
conv(a:INT,b:INT):INT == a+ (b-1)*w;
counter := 0
k == counter
c:Integer
coeflist:List Polynomial Integer:=[-1,a1,a2,a3,a4,a6,a7,a8,a9,a10]
--we make a matrix of the correct size
ma:Matrix Polynomial Integer := new(max{numeqs,h*w},h*w,0);
--this function labels the columns of our matrix
for i in 1..w repeat
  if i+step>w then leave
  for j in 1..h repeat
    counter := counter +1
    output counter
    for t in 0..step repeat
      c := conv(i+step-t,j)
```

```
      ma(k,c):= coeflist(t+1)
for i in 1..w repeat
  for j in 1..h repeat
    if j+step>h then leave
    counter := counter +1
    output counter
    for t in 0..step repeat
      c:=conv(i,j+step-t)
      ma(k,c):= coeflist(t+1)
for i in 1..w repeat
  if i+step>w then leave
  for j in 1..h repeat
    if j+step>h then leave
    counter:= counter +1
    output counter
    for t in 0..step repeat
      c:=conv(i+t,j+step-t)
      ma(k,c):=coeflist(t+1)
for i in 1..w repeat
  if i+step-1>w then leave
  for j in 1..h repeat
    if j+step-1>h then leave
    counter := counter +1
    output counter
    for s in 1..step repeat
      for t in 1..s repeat
        c:= conv(i+t-1,j+s-t)
        ma(k,c):=coeflist(step-s+1)
st:INT
summary:=["equations",nrows(ma),"variables",ncols(ma)]@List(Any);
```

```
output summary;
for l in 1..10 repeat
  for m in 1..10 repeat
    mint:=map(x+->eval(x,[a1,a2],[l,m]),ma);
    remint:= rowEchelon(mint);
    st:=1
    for i in 1..ncols(remint) repeat st:=st*remint(i,i)
    if st>0 then fst:=factor(st) else fst := st;
    datastore:=[[l,m],fst]@List(Any)
    output datastore
```

" The code called cabstep3 "

```
)set message time on
)spool cabstep3.out
step:= 3
--input width of rectangle
w := 6
--input height of rectangle
h := 5
--program calculates number of equations generated
horiz:= max {0,h*(w-step)}
vert := max {0,w*(h-step)}
diag := max {0,(h-step)*(w-step)}
triang := max{0,(h-step+1)*(w-step+1)}
numeqs := horiz + vert + diag + triang
conv(a:INT,b:INT):INT == a+ (b-1)*w;
counter := 0
k == counter
c:Integer
coeflist:List Polynomial Integer:=[-1,a1,a2,a3,a4,a6,a7,a8,a9,a10]
--we make a matrix of the correct size
ma:Matrix Polynomial Integer := new(max{numeqs,h*w},h*w,0);
--this function labels the columns of our matrix
for i in 1..w repeat
  if i+step>w then leave
  for j in 1..h repeat
    counter := counter +1
    output counter
    for t in 0..step repeat
      c := conv(i+step-t,j)
      ma(k,c):= coeflist(t+1)
```

```
for i in 1..w repeat
  for j in 1..h repeat
    if j+step>h then leave
    counter := counter +1
    output counter
    for t in 0..step repeat
      c:=conv(i,j+step-t)
      ma(k,c):= coeflist(t+1)
for i in 1..w repeat
  if i+step>w then leave
  for j in 1..h repeat
    if j+step>h then leave
    counter:= counter +1
    output counter
    for t in 0..step repeat
      c:=conv(i+t,j+step-t)
      ma(k,c):=coeflist(t+1)
for i in 1..w repeat
  if i+step-1>w then leave
  for j in 1..h repeat
    if j+step-1>h then leave
    counter := counter +1
    output counter
    for s in 1..step repeat
      for t in 1..s repeat
        c:= conv(i+t-1,j+s-t)
        ma(k,c):=coeflist(step-s+1)
st:INT
summary:=["equations",nrows(ma),"variables",ncols(ma)]@List(Any);
output summary;
```

```
for l in 1..5 repeat
  for m in 1..5 repeat
    for n in 1..5 repeat
        mint:=map(x+->eval(x,[a1,a2,a3],[l,m,n]),ma);
        remint:= rowEchelon(mint);
        st:=1
        for i in 1..ncols(remint) repeat st:=st*remint(i,i)
        if st>0 then fst:=factor(st) else fst := st;
        datastore:=[[l,m,n],fst]@List(Any)
        output datastore
```

“the code called xabcstep2”

```
)set message time on
)spool xabcstep2.out
-- Declare the type of some  variables.
f:Integer
remint : Matrix Integer
store  : Integer
h11: Matrix Polynomial Integer
h12: Matrix Polynomial Integer
h21: Matrix Polynomial Integer
h22: Matrix Polynomial Integer
-- Set up the matrix c, and build matrixes aa (=A) and bb (= B).
cc:Matrix Polynomial Integer:= new(2,2,0);
cc(1,2):= 1;
cc(2,1):= 1;
cc(2,2):= m;
aa := 1 -cc;
-- Now build 2 by 1 matrices containing our 3 "unknowns".
a:Matrix Polynomial Integer:= new(2,1,0);
a(1,1):= a1;
a(2,1):= a2;
-- Now build the 4 by 4 matrix of vectors of unknowns.
m31 := a;
m32 := aa*m31;
m21 := aa*m31;
m11 := aa*m21;
m22 := aa*m21;
m33 := aa*m32;
m12 := aa*m11;
m13 := aa*m12;
```

```
m23 := aa*m22;
-- We get 1 diagonal "vector equations"
-- which are really 2 "ordinary" equations.
di31: Matrix Polynomial Integer
di31 := cc**2*m31 - l*cc*m22 - m*m13;
-- We now search for bad primes while varying the coefficients of our
-- recurrence.
for ii in 1..10 repeat
  for jj in 1..10 repeat
       st := map(x+->eval(x,[l,m],[ii,jj]),di31);
       co : Matrix Integer := new(2,2,0)
       for i in 1..2 repeat
           k := st(i,1)
           f := differentiate(k,a2)
           co(i,1) := f
           f := differentiate(k,a1)
           co(i,2) := f
-- We have built the matrix of the equations. we now look for
-- primes at which this matrix will fail to have full rank.
       remint:= rowEchelon(co);
       store:=1
       for i in 1..ncols(remint) repeat store:=store*remint(i,i)
       if store>0 then fst:=factor(store) else fst := store;
           datastore:=[[ii,jj],fst]@List(Any)
           output datastore
```

“the code called xabcstep3”

```
)set message time on
)spool xabcstep3.out
-- Declare the type of some  variables.
f:Integer
remint : Matrix Integer
store  : Integer
h11: Matrix Polynomial Integer
h12: Matrix Polynomial Integer
h21: Matrix Polynomial Integer
h22: Matrix Polynomial Integer
-- Set up the matrix c, and build matrixes aa (=A) and bb (= B).
cc:Matrix Polynomial Integer:= new(3,3,0);
cc(1,2):= 1;
cc(2,3):= 1;
cc(3,1):= n;
cc(3,2):= m;
cc(3,3):= l;
aa := m + l*cc -cc**2;
bb := l - cc;
-- Now build 3 by 1 matrices containing our 12 "unknowns".
a:Matrix Polynomial Integer:= new(3,1,0);
a(1,1):= a1;
a(2,1):= a2;
a(3,1):= a3;
b:Matrix Polynomial Integer:= new(3,1,0);
b(1,1):= b1;
b(2,1):= b2;
b(3,1):= b3;
c:Matrix Polynomial Integer:= new(3,1,0);
```

```
c(1,1):= c1;
c(2,1):= c2;
c(3,1):= c3;
d:Matrix Polynomial Integer:= new(3,1,0);
d(1,1):= d1;
d(2,1):= d2;
d(3,1):= d3;
-- Now build the 5 by 5 matrix of vectors of unknowns.
m51 := a;
m52 := b;
m41 := c;
m42 := d;
m31 := aa*m51 + bb*m41;
m21 := aa*m41 + bb*m31;
m11 := aa*m31 + bb*m21;
m32 := aa*m52 + bb*m42;
m22 := aa*m42 + bb*m32;
m12 := aa*m32 + bb*m22;
m13 := aa*m11 + bb*m12;
m14 := aa*m12 + bb*m13;
m15 := aa*m13 + bb*m14;
m23 := aa*m21 + bb*m22;
m24 := aa*m22 + bb*m23;
m25 := aa*m23 + bb*m24;
m33 := aa*m31 + bb*m32;
m34 := aa*m32 + bb*m33;
m35 := aa*m33 + bb*m34;
m43 := aa*m41 + bb*m42;
m44 := aa*m42 + bb*m43;
m45 := aa*m43 + bb*m44;
```

```
m53 := aa*m51 + bb*m52;
m54 := aa*m52 + bb*m53;
m55 := aa*m53 + bb*m54;
-- We get 4 diagonal "vector equations"
-- which are really 12 "ordinary" equations.
di51: Matrix Polynomial Integer
di42: Matrix Polynomial Integer
di41: Matrix Polynomial Integer
di52: Matrix Polynomial Integer
di51 := cc**3*m51 - l*cc**2*m42 - m*cc*m33 - n*m24;
di42 := cc**3*m42 - l*cc**2*m33 - m*cc*m24 - n*m15;
di41 := cc**3*m41 - l*cc**2*m32 - m*cc*m23 - n*m14;
di52 := cc**3*m52 - l*cc**2*m43 - m*cc*m34 - n*m25;
-- We now search for bad primes while varying the coefficients of our
-- recurrence.
for ii in 1..5 repeat
  for jj in 1..5 repeat
    for kk in 1..5 repeat
       h11 := map(x+->eval(x,[l,m,n],[ii,jj,kk]),di51);
       h12 := map(x+->eval(x,[l,m,n],[ii,jj,kk]),di42);
       h21 := map(x+->eval(x,[l,m,n],[ii,jj,kk]),di41);
       h22 := map(x+->eval(x,[l,m,n],[ii,jj,kk]),di52);
       st1 := vertConcat(h11,h12);
       st := vertConcat(st1,st2);
       co : Matrix Integer := new(12,12,0)
       for i in 1..12 repeat
           k :=st(i,1);
           f := differentiate(k,d3)
           co(i,1) := f
           f := differentiate(k,d2)
```

```
          co(i,2) := f
          f := differentiate(k,d1)
          co(i,3) :=f
          f := differentiate(k,c3)
          co(i,4) :=f
          f := differentiate(k,c2)
          co(i,5) :=f
          f := differentiate(k,c1)
          co(i,6) :=f
          f := differentiate(k,b3)
          co(i,7) :=f
          f := differentiate(k,b2)
          co(i,8) :=f
          f := differentiate(k,b1)
          co(i,9) :=f
          f := differentiate(k,a3)
          co(i,10) :=f
          f := differentiate(k,a2)
          co(i,11) :=f
          f := differentiate(k,a1)
          co(i,12) :=f
-- We have built the matrix of the equations. we now look for
-- primes at which this matrix will fail to have full rank.
      remint:= rowEchelon(co);
      store:=1
      for i in 1..ncols(remint) repeat store:=store*remint(i,i)
      if store>0 then fst:=factor(store) else fst := store;
          datastore:=[[ii,jj,kk],fst]@List(Any)
          output datastore
```

“the code called cayley ”

```
procedure ort(a,b,c,l,m,n;b,c,d);
        d = a^n*b^m*c^l;
end;


for l = 1 to 5 do
  for m = 1 to 5 do
    for n = 1 to 5 do
      for p = 5 to 20 do
        catch = 2;
        if not prime(p) then loop; end;
        f = fgrank(3);
        g = pquotient(f,p,3;explaw=p);
        n1 = normal closure(g,<g.1,(g.3,g.2,g.3)>);
        h1,f1 = g/n1;
        n2  = normal closure(g,<g.3,(g.2,g.1,g.2)>);
        h2,f2 = g/n2;
        new = n1 meet n2;
        a = g.1;
        b = g.2;
        c = g.3;
        while not order(new) eq 1 do
                ort(a,b,c,l,m,n;a,b,c);
                n1 = normal closure(g,<a,(c,b,c)>);
                n2 = normal closure(g,<c,(b,a,b)>);
                new = new meet (n1 meet n2);
                if order(new) eq 1 then
                        print l,m,n,'is fine at the prime ',p;
                        break;
                end;
```

```
                if catch eq order(new) then
                        print l,m,n,'is dodgy at the prime ',p;
                        break;
                end;
                catch = order(new);
        end;
      end;
    end;
  end;
end;
```

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